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Probability and Statistics for Economists
Chapter 3 Random Variables and Univariate Probability Distributions

1. Seven balls are distributed randomly into seven cells. Let X_i = the number of cells containing exactly i balls. What is the probability distribution of X_3? (That is, find $P(X = x)$ for every possible x.)
Solution:

There are 7^7 equally likely sample points. The possible values of X_3 are 0, 1 and 2. Only the pattern 331 (3 balls in one cell, 3 balls in another cell and 1 ball in a third cell) yields $X_3 = 2$. The number of sample points with this pattern is $\binom{7}{2}\binom{7}{3}\binom{4}{3}5 = 14{,}700$. So $P(X_3 = 2) = 14,700/7^7 \approx .0178$. There are 4 patterns that yield $X_3 = 1$. The number of sample points that give each of these patterns is given below.

pattern	number of sample points
34	$7\binom{7}{3}6 = 1470$
322	$7\binom{7}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 22050$
3211	$7\binom{7}{3}6\binom{4}{2}\binom{5}{2}2! = 176400$
31111	$7\binom{7}{3}\binom{6}{4}4! = 88200$
	sum $= 288120$

So $P(X_3 = 1) = 288120/7^7 \approx 0.3498$. The number of sample points that yield $X_3 = 0$ is $7^7 - 288120 - 14700 = 520723$, and $P(X_3 = 0) = 520723/7^7 \approx 0.6322$.

2. Prove that the following functions are cumulative distribution functions (CDF's):

(1) $\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), x \in (-\infty, +\infty)$;

(2) $(1 + e^{-x})^{-1}, x \in (-\infty, +\infty)$;

(3) $e^{-e^{-x}}, x \in (-\infty, +\infty)$;

(4) $1 - e^{-x}, x \in (0, +\infty)$, and $0, x \leq 0$.

Solution:

All of the functions are continuous, hence right-continuous. Thus we only need to check the limit, and that they are nondecreasing.

(a) $\lim\limits_{x\to-\infty} \frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2}+\frac{1}{\pi}\left(\frac{-\pi}{2}\right) = 0$, $\lim\limits_{x\to\infty} \frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2}+\frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1$, and $\frac{d}{dx}\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)\right) = \frac{1}{1+x^2} > 0$

(b) $\lim\limits_{x\to-\infty} (1 + e^{-x})^{-1} = 0$, $\lim\limits_{x\to\infty} (1 + e^{-x})^{-1} = 1$, $\frac{d}{dx}(1 + e^{-x})^{-1} = \frac{e^{-x}}{(1+e^{-x})^2} > 0$

(c) $\lim\limits_{x\to-\infty} e^{-e^{-x}} = 0$, $\lim\limits_{x\to\infty} e^{-e^{-x}} = 1$, $\frac{d}{dx}e^{-e^{-x}} = e^{-x}e^{-e^{-x}} > 0$

(d) $\lim\limits_{x\to-\infty} (1 - e^{-x}) = 0$, $\lim\limits_{x\to\infty} (1 - e^{-x}) = 1$, $\frac{d}{dx}(1 - e^{-x}) = e^{-x} > 0$

3. A CDF F_X is stochastically greater than a CDF F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X \sim F_X$ and $Y \sim F_Y$, then

$$\begin{aligned} P(X > t) &\geq P(X > t) \quad \text{for every } t \\ P(X > t) &> P(X > t) \quad \text{for some } t, \end{aligned}$$

that is, X tends to be bigger than Y.
Solution:

By definition of CDF function, $F_X(t) = P(X \leq t) = 1 - P(X > t)$. Thus $F_X(t) \leq F_Y(t) \iff 1 - P(X > t) \leq 1 - P(Y > t) \Leftrightarrow P(X > t) \geq P(Y > t)$ for every t.

And $F_X(t) < F_Y(t) \iff 1 - P(X > t) < 1 - P(Y > t) \Leftrightarrow P(X > t) > P(Y > t)$ for some t.

4. Suppose $X = X_1$ with probability p and $X = X_2$ with probability $1 - p$, where $p \in (0, 1)$, X_1 and X_2 are random variables with CDF's $F_1(x)$ and $F_2(x)$ respectively. Find the CDF of X.

Solution:

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) \\ &= p \times \Pr(X_1 \leq x) + (1-p) \times \Pr(X_2 \leq x) \\ &= pF_1(x) + (1-p)F_2(x) \end{aligned}$$

5. Let $f(x) = \frac{c}{x}$ for $x = 1, 2, \cdots$ and c is a constant. Can you find a finite value for constant c so that $f(x)$ is a valid PMF? If yes, give the value of c. Otherwise, explain why not.

Solution:

Yes. Proof skipped.

6. An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given that the cumulative distribution of T, the number of years to maturity for a randomly selected bond is

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \leq t < 3, \\ \frac{1}{2}, & 3 \leq t < 5, \\ \frac{3}{4}, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$$

Find (1) $P(T = 5)$; (2) $P(T > 3)$; and (3) $P(1.4 < T < 6)$. Give your reasoning.

Solution:

(a)

$$\begin{aligned} P(T &= 5) = \lim_{\delta \to 0^+} \left[1 - \Pr\left(T > 5 + \delta\right) - \Pr\left(T < 5 - \delta\right)\right] \\ &= \lim_{\delta \to 0^+} \left[1 - \left(1 - \Pr\left(T \leq 5 + \delta\right)\right) - \Pr\left(T < 5 - \delta\right)\right] \\ &= \lim_{\delta \to 0^+} \left[F(5 + \delta) - F(5 - \delta)\right] \\ &= \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \end{aligned}$$

(b) $P(T > 3) = 1 - P(T \leq 3) = 1 - \frac{1}{2} = \frac{1}{2}$

(c) $P(1.4 < T < 6) = P(T < 6) - P(T \leq 1.4) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

(d)

$$\begin{aligned} f(t) &= \frac{1}{4} && \text{if t=1} \\ &= \frac{1}{4} && \text{if t=3} \\ &= \frac{1}{4} && \text{if t=5} \\ &= \frac{1}{4} && \text{if t=7} \\ &= 0 && \text{otherwise.} \end{aligned}$$

7. For each of the following, determine the value of c that makes $f(x)$ a PDF:

(1) $f(x) = c \sin x, 0 < x < \frac{\pi}{2}$;

(2) $f(x) = ce^{-|x|}, -\infty < x < \infty$.

Solution:

(1) $\int_0^{\pi/2} \sin x dx = 1$. Thus, $c = 1$.

(2) $\int_{-\infty}^{\infty} e^{-|x|} dx = \int_{-\infty}^{0} e^{x} dx + \int_0^{\infty} e^{-x} dx = 1 + 1 = 2$. Thus, $c = 0.5$.

8. Suppose X has the geometric PMF $f_X(x) = \frac{1}{3}(\frac{2}{3})^x, x = 0, 1, 2, ...$ Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its PMF.

Solution:

$P(Y = y) = P(\frac{X}{X+1} = y) = P(X = \frac{y}{1-y}) = \frac{1}{3}(\frac{2}{3})^{y/(1-y)}$, where $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{x}{x+1}, \cdots$ and 0 elsewhere.

9. Let X have the PDF

$$f_X(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, 0 < x < \infty, \beta > 0.$$

Verify that $f_X(x)$ is indeed a PDF. [Hint: you may use the property that the integral of the pdf of a normal random variable is 1.]

Solution:

By definition of PDF, $f(x)$ must satisfy the following properties:

- $f(x) \geq 0$
  Thus, $\beta > 0$ for all $-\infty < x < \infty$
- $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\begin{aligned} LHS &= \frac{4}{\beta^3\sqrt{\pi}} \int_{-\infty}^{\infty} (\frac{x}{\beta})^2 e^{-(\frac{x}{\beta})^2} \beta^2 \beta \frac{1}{2} (\frac{x}{\beta})^{-1} d(\frac{x}{\beta})^2 \\ &= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^{\frac{3}{2}-1} e^{-z} dz \\ &= \frac{2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) \\ &= 1 \end{aligned} \tag{1}$$

Since $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(1) = \frac{\sqrt{\pi}}{2}$

10. Let $f(x) = \frac{c}{x}$ for $x = 1, 2, \cdots$ and c is a constant. Can you find a finite value for constant c so that $f(x)$ is a valid PMF? If yes, give the value of c. Otherwise, explain why not.

Solution:

Since $-1 < 1 + 2sinx < 3$ where $-\pi < x < \pi$, there is no possible value of c ensuring that the pdf $f(x) \geq 0$ for $-\pi < x < \pi$.

11. Check for what value(s) of k that the following function can be a PDF:

$$f(x) = \begin{cases} \frac{1}{2} + kx, & -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Give your reasoning.

Solution:

- If $c \geq 0$, then by definition of PDF, $f(x) = \frac{1}{2} + cx \geq 0 \rightarrow 0 \leq c \leq \frac{1}{2}$

- If $c \le 0$, then by definition of PDF, $f(x)\frac{1}{2} + cx \ge 0 \to -\frac{1}{2} \le c \le 0$

12. Suppose $f_X(x)$ and $f_Y(y)$ are two PDF's. Define $g(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy$. Is $g(z)$ a PDF? Explain.

Solution:

Easy to see $f(z) \ge 0$. To show $\int_{-\infty}^{\infty} f(z)dz = 1$, notice that

$$\int_{-\infty}^{\infty} f(z)dz = \sum_{i=1}^{k} \int_{-\infty}^{\infty} y_i^{-1} f_X(z/y_i) f_Y(y_i) dz = \sum_{i=1}^{k} f_Y(y_i) = 1$$

13. In each of the following, find the PDF of Y and show that the PDF integrates to 1:

(1) $f_X(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$; $Y = |X|^3$;

(2) $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$.

Solution:

(1)The support of Y is $\Omega_Y = \{y \in R : y > 0\}$. We apply the CDF approach to figure out the density function of $Y = |X|^3$

$$\begin{aligned} F_Y(y) = P(Y \le y) &= P(|X|^3 \le y) \\ &= P(|X| \le y^{\frac{1}{3}}) \\ &= P(-y^{\frac{1}{3}} \le X \le y^{\frac{1}{3}}) \\ &= F_X(y^{\frac{1}{3}}) - F_X(-y^{\frac{1}{3}}) \end{aligned} \tag{2}$$

$$\begin{aligned} f_Y(y) = \frac{\partial F_Y(y)}{\partial y} &= f_X(y^{\frac{1}{3}})\frac{1}{3}y^{-\frac{2}{3}} - f_X(-y^{\frac{1}{3}})\frac{1}{3}(-1)y^{-\frac{2}{3}} \\ &= \frac{1}{3}e^{-y^{\frac{1}{3}}}y^{-\frac{2}{3}} \end{aligned} \tag{3}$$

Thus, the PDF of random variable Y is:

$$f_Y(y) = \begin{cases} \frac{1}{3}e^{-y^{\frac{1}{3}}}y^{-\frac{2}{3}} & y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

To test the integral of PDF $f_Y(y)$ over the support Ω_Y, we have the following result:

$$\begin{aligned} \int_0^{\infty} f_Y(y)dy &= \int_0^{\infty} e^{-y^{\frac{1}{3}}} dy^{\frac{1}{3}} \\ &= -e^{z}|_0^{\infty} \quad \text{by changing variables, define } z = y^{\frac{1}{3}} \\ &= 1 \end{aligned} \tag{5}$$

(2) The support of Y is $\Omega_Y = \{y \in R : 0 < y < 1\}$. We apply the CDF approach to figure out the density function of $Y = 1 - X^2$

$$\begin{aligned} F_Y(y) &= P(1 - X^2 \le y) \\ &= P(X^2 \ge 1 - y) \\ &= P(X \ge \sqrt{(1-y)}) + P(X \le -\sqrt{1-y}) \\ &= 1 - F_X(\sqrt{1-y}) + F_X(-\sqrt{1-y}) \end{aligned} \tag{6}$$

$$
\begin{aligned}
f_Y(y) &= \frac{\partial F_Y(y)}{\partial y} \\
&= f_X(\sqrt{1-y})\frac{1}{2}(1-y)^{-\frac{1}{2}} + f_X(-\sqrt{1-y})\frac{1}{2}(1-y)^{-\frac{1}{2}} \\
&= \frac{3}{16}(1-y)^{-\frac{1}{2}}(1+1-y+2\sqrt{(1-y)}+1+1-y-2\sqrt{(1-y)}) \\
&= \frac{3}{8}(1-y)^{-\frac{1}{2}} + \frac{3}{8}(1-y)^{\frac{1}{2}}
\end{aligned} \tag{7}
$$

Thus, the PDF of random variable Y is:

$$
f_Y(y) = \begin{cases} \frac{3}{8}(1-y)^{-\frac{1}{2}} + \frac{3}{8}(1-y)^{\frac{1}{2}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}
$$

To test whether or not the integral of the PDF function equals to 1, we have the following result:

$$
\begin{aligned}
\int_0^1 f_Y(y)dy &= \frac{3}{8}\int_0^1 (1-y)^{-\frac{1}{2}}dy + \frac{3}{8}\int_0^1 (1-y)^{\frac{1}{2}}dy \\
&= \frac{3}{8}\int_0^1 z^{-\frac{1}{2}}dz + \frac{3}{8}\int_0^1 z^{\frac{1}{2}}dz \quad \text{by changing variables, z=1-y} \\
&= \frac{3}{4} + \frac{1}{4} \\
&= 1
\end{aligned} \tag{9}
$$

14. Let X have PDF $f_X(x) = \frac{2}{9}(x+1)$, $-1 \leq x \leq 2$. Find the PDF of $Y = X^2$.
Solution:

See section notes or previous homework.

15.$f_X(x) = \frac{1}{2}$ for $0 < x < 2$. Find the PDF of $Y = X(2-X)$.
Solution:

The support of random variable Y is $\Omega = \{y \in R : 0 < y < 1\}$, since the mapping between X and Y is not monotone, we apply the CDF approach here.

$$
\begin{aligned}
F_Y(y) &= P(X(2-X) \leq y) \\
&= P(X^2 - 2X - y \geq 0) \quad \text{since } 0 < y < 1,\ \Delta = \sqrt{1-y} \text{ is well-defined and it guarantees real solutions} \\
&= P(X \geq 1 + \sqrt{1-y}) + P(X \leq 1 - \sqrt{1-y}) \\
&= 1 - F(1+\sqrt{1-y}) + F(1-\sqrt{1-y})
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
f_Y(y) &= \frac{\partial F_Y(y)}{\partial y} \\
&= \frac{1}{2}(1-y)^{-\frac{1}{2}}
\end{aligned} \tag{11}
$$

Thus, the PDF of the random variable Y is:

$$
f_Y(y) = \begin{cases} \frac{1}{2}(1-y)^{-\frac{1}{2}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{12}
$$

16. (1) Let X be a continuous, nonnegative random variable, i.e., $f(x) = 0$ for $x < 0$. Show that

$E(X) = \int_0^\infty [1 - F_X(x)]\, dx$, where $F_X(x)$ is the CDF of X;

(2) Let X be a discrete random variable whose range is nonnegative integers. Show $E(X) = \sum_{k=0}^{\infty}[1 - F_X(k)]$, where $F_X(k) = P(X \leq k)$. Compare this with Part (1).

Solution:

(1)

$$\begin{aligned}
\int_0^\infty [1 - F_X(x)]dx &= \int_0^\infty P(X > x)dx = \int_0^\infty \left[\int_x^\infty f_X(y)dy\right] dx \\
&= \int_0^\infty \left[\int_0^y f_X(y)dx\right] dy = \int_0^\infty \left[f_X(y) \int_0^y dx\right] dy \\
&= \int_0^\infty [f_X(y)y]\, dy = E(X)
\end{aligned}$$

(2)

$$\begin{aligned}
\sum_{k=0}^{\infty} [1 - F_X(k)] &= \sum_{k=0}^{\infty} P(X > k) \\
&= P(X = 1) + P(X = 2) + P(X = 3) + \ldots \\
+P(X &= 2) + P(X = 3) + \ldots + P(X = 3) + \ldots \\
&= \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} kP(X = k) + 0P(X = 0) \\
&= E(X)
\end{aligned}$$

17. Show that if X is a continuous random variables, then

$$\min_a E\,|X - a| = E\,|X - m|\,,$$

where m is the median of X.

Solution:

Let m be the median. By definition, $\int_{-\infty}^m f(x)dx = \frac{1}{2}$. Cauchy distribution Cauchy(0,1)

$$\int_{-\infty}^m \frac{1}{\pi(1 + x^2)} dx = \frac{1}{\pi} \arctan x|_{-\infty}^m = \frac{1}{\pi}(\arctan m + \frac{\pi}{2}) = \frac{1}{2}$$

So $m = 0$. That is , the median of this Cauchy distributionis 0. ($f(x)$ is symmetric about 0, thus the median and mean is 0).

18. Let X have the PDF

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, 0 < x < \infty, \beta > 0$$

Find $E(X)$ and $\mathrm{Var}(X)$.

Solution:

(1)

$$
\begin{aligned}
E(X) &= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x x^2 e^{\frac{-x^2}{\beta^2}} dx \\
&= \frac{4\beta^4}{\beta^3\sqrt{\pi}} \int_0^\infty \frac{1}{2}\frac{x^2}{\beta^2} e^{-\frac{x^2}{\beta^2}} d\frac{x^2}{\beta^2} \\
&= \frac{2\beta}{\sqrt{\pi}} \int_0^\infty z e^{-z} dz \quad \text{by changing variables, } z = \frac{x^2}{\beta^2} \\
&= \frac{2\beta}{\sqrt{\pi}} \Gamma(2) \\
&= \frac{2\beta}{\sqrt{\pi}}
\end{aligned} \tag{13}
$$

(2)

$$
\begin{aligned}
E(X^2) &= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x x^3 e^{-\frac{x^2}{\beta^2}} dx \\
&= \frac{4\beta^5}{\beta^3\sqrt{\pi}} \int_0^\infty \frac{1}{2}\frac{x^3}{\beta^3} e^{-\frac{x^2}{\beta^2}} d\frac{x^2}{\beta^2} \\
&= \frac{2\beta^2}{\sqrt{\pi}} \int_0^\infty z^{\frac{3}{2}} e^{-z} dz \quad \text{by changing variable, } z = \frac{x^2}{\beta^2} \\
&= \frac{2\beta^2}{\sqrt{\pi}} \Gamma(\frac{3}{2}+1) \\
&= \frac{2\beta^2}{\sqrt{\pi}} \frac{3}{2} \Gamma(\frac{1}{2}+1) \\
&= \frac{2\beta^2}{\sqrt{\pi}} \frac{3}{4} \sqrt{\pi} \quad \text{since } \Gamma(\frac{1}{2} = \sqrt{\pi}) \\
&= \frac{3}{2}\beta^2
\end{aligned} \tag{14}
$$

Thus, the variance of random variable X is

$$
Var(X) = EX^2 - (EX)^2 = \frac{3}{2}\beta^2 - \frac{4}{\pi}\beta^2 \tag{15}
$$

19. Let $f(x)$ be a PDF, and let a be a number such that, for all $\epsilon > 0$, $f(a+\epsilon) = f(a-\epsilon)$. Such a pdf is said to be symmetric about the point a.

(1) Give three examples of symmetric PDF's;

(2) Show that if $X \sim f(x)$, symmetric, then the median of X is the number a;

(3) Show that if $X \sim f(x)$, symmetric, and $E(X)$ exists,then $E(X) = a$.

Solution:

(1)Three examples of symmetric pdfs: $U[0,1]$: symmetric about $\frac{1}{2}$; Cauchy(0,1): $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in R$, symmetric about 0; $N(0,1)$: symmetric about 0.

(2)To show $\int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx = \frac{1}{2}$:

$$\begin{aligned}
\int_{-\infty}^{a} f(x)dx &= \int_{0}^{\infty} f(a-\varepsilon)d\varepsilon \text{ letting } x = a-\varepsilon \\
&= \int_{0}^{\infty} f(a+\varepsilon)d\varepsilon \text{ by symmetry of } f(\cdot) \\
&= \int_{a}^{\infty} f(x)dx \text{ letting } x = a+\varepsilon
\end{aligned}$$

And $\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = 1$. So $\int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx = \frac{1}{2}$. Hence a is the median.

(3)To show $EX = a$

$$\begin{aligned}
EX - a &= E(X-a) = \int_{-\infty}^{\infty} (x-a)f(x)dx \\
&= \int_{-\infty}^{a} (x-a)f(x)dx + \int_{a}^{\infty} (x-a)f(x)dx \\
&= \int_{0}^{\infty} (-\varepsilon)f(a-\varepsilon)d\varepsilon + \int_{0}^{\infty} \varepsilon f(a+\varepsilon)d\varepsilon \\
&= -\int_{0}^{\infty} \varepsilon f(a+\varepsilon)d\varepsilon + \int_{0}^{\infty} \varepsilon f(a+\varepsilon)d\varepsilon = 0
\end{aligned}$$

20. A random variable X is said to have a two-piece normal distribution with parameter $\alpha, \sigma_1, \sigma_2$ if its PDF

$$f_X(x) = \begin{cases} Ae^{-(x-\alpha)^2/(2\sigma_1^2)}, & x \le \alpha, \\ \\ Ae^{-(x-\alpha)^2/(2\sigma_2^2)}, & x > \alpha. \end{cases}$$

Find: (1) the constant A; (2) the mean of X; (3) the variance of X.

Solution:

(1) for $\forall x \in R$, $f_X(x) \ge 0$ iff $A \ge 0$; There exists a A such that $\int_{-\infty}^{+\infty} f_X(x)dx = 1$;

$$\begin{aligned}
LHS &= A\int_{-\infty}^{\alpha} e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} dx + A\int_{\alpha}^{\infty} e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} dx \\
&= A(\sqrt{2\pi}\sigma_1)\int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} dx + A(\sqrt{2\pi}\sigma_2)\int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} dx \\
&= A(\sqrt{2\pi}\sigma_1)\frac{1}{2} + A(\sqrt{2\pi}\sigma_2)\frac{1}{2} \\
&= A\sqrt{\frac{\pi}{2}}(\sigma_1 + \sigma_2)
\end{aligned}$$

Thus, $A = \frac{\sqrt{\frac{2}{\pi}}}{\sigma_1 + \sigma_2}$, which also satisfies the non-negativity condition above. **Trick: Apply the property of symmetric functions.**

(2) Find the mean of X By definition

$$
\begin{aligned}
EX &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= A \int_{-\infty}^{\alpha} x e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} dx + A \int_{\alpha}^{+\infty} x e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} dx \\
&= A \int_{-\infty}^{\alpha} (x-\alpha) e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} dx + A \int_{\alpha}^{\infty} (x-\alpha) e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} dx \\
&+ A\alpha \int_{-\infty}^{\alpha} e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} dx + A\alpha \int_{\alpha}^{+\infty} e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} dx \quad \text{(by adding and subtracting)} \\
&= -A\sigma_1^2 e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} |_{-\infty}^{\alpha} - A\sigma_2^2 e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} |_{\alpha}^{+\infty} + A\alpha\sqrt{2\pi}\sigma_1 \frac{1}{2} + A\alpha\sqrt{2\pi}\sigma_2 \frac{1}{2} \\
&= \sqrt{\frac{2}{\pi}} (\sigma_2 - \sigma_1) + \alpha
\end{aligned}
$$

Trick: by using adding and subtracting, changing variables and property of symmetric functions

(3) Find the variance of X By definition, $Var(X) = EX^2 - (EX)^2$, we start by computing the second moment of r.v. X:

$$
\begin{aligned}
EX^2 &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\
&= A \int_{-\infty}^{\alpha} x(x-\alpha) e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} dx + A \int_{\alpha}^{+\infty} x(x-\alpha) e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} dx \\
&+ A\alpha \int_{-\infty}^{\alpha} x e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}} dx + A\alpha \int_{\alpha}^{\infty} x e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}} dx \\
&= -A\sigma_1^2 \int_{-\infty}^{\alpha} x d(e^{-\frac{(x-\alpha)^2}{2\sigma_1^2}}) - A\sigma_2^2 \int_{\alpha}^{\infty} x d(e^{-\frac{(x-\alpha)^2}{2\sigma_2^2}}) + \alpha EX \\
&= -A\sigma_1^2\alpha + A\sigma_2^2\alpha + A\sigma_1^3\sqrt{2\pi}\frac{1}{2} + A\sigma_2^3\sqrt{2\pi}\frac{1}{2} + \alpha EX
\end{aligned}
$$

Thus, Plugging EX^2 and EX back into the equation, we can compute the variance of X **Trick: Applying changing variables, integrating by part and property of symmetric functions**

21. Suppose a discrete random variable X has the following distribution

$$P_X(x) = (1-\gamma)\gamma^x, x = 0, 1, \cdots,$$

where γ is a fixed parameter and $0 < \gamma < 1$. Find: (1) the MGF of X; (2) the mean and the variance. [Hint: use the formula $\sum_{x=0}^{\infty} a^x = \frac{1}{1-a}$ for $|a| < 1$.]

Solution:

(1)

$$\begin{aligned} M_X(t) &= Ee^{tx} = \sum_{x=0}^{\infty}(1-r)r^x e^{tx} \\ &= (1-r)\sum_{x=0}^{\infty}(re^t)^x \end{aligned}$$

When there is a small number $\epsilon(\leq \ln\frac{1}{r}) > 0$ such that $\forall t \in (-\epsilon, \epsilon)$, $|re^t| < 1$. Then $\sum_{x=0}^{\infty}(re^t)^x = \frac{1}{1-re^t}$. So $M_X(t) = \frac{1-r}{1-re^t}$.

(2)

$$\begin{aligned} EX &= M'_X(0) = (1-r)\frac{re^t}{(1-re^t)^2}|_{t=0} = \frac{(1-r)r}{(1-r)^2} = \frac{r}{1-r} \\ EX^2 &= M''_X(0) = \frac{(1-r)(re^t + (re^t)^2)}{(1-re^t)^3}|_{t=0} = \frac{r(1+r)}{(1-r)^2} \\ Var(X) &= EX^2 - (EX)^2 = \frac{r}{(1-r)^2} \end{aligned}$$

22. Suppose that a discrete random variable X has variance $\sigma_X^2 = \frac{1}{2}$ and moment generating function

$$M_X(t) = a + b(e^{-t} + e^t), \quad -\infty < t < \infty.$$

Find the PMF $f_X(x)$ of X and justify your answer.

Solution:

Following the same methods in question 5, it is easy to find that $a = \frac{1}{2}$ and $b = \frac{1}{4}$. And the unique PDF of X is:

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } X = 0 \\ \frac{1}{4} & \text{if } X = -1 \\ \frac{1}{4} & \text{if } X = 1 \\ 0 & \text{otherwise} \end{cases}$$

23. Let X and Y be two discrete random variables with the identical set of possible values $\Omega = \{a_1, a_2, ..., a_n\}$, where the a_i's are n different real numbers. Show that if $E(X^k) = E(Y^k)$ for $k = 1, 2, ..., n-1$, then X and Y are identically distributed; that is, $P(X = t) = P(Y = t)$ for $t \in \{a_1, ..., a_n\}$.

Solution: Define a $n \times n$ matrix A such that each column is $A_i = [1, a_i, a_i^2, ..., a_i^{n-1}]'$. Denote $n \times 1$ vector b to be $[f_X(a_1) - f_Y(a_1), f_X(a_2) - f_Y(a_2), ..., f_X(a_n) - f_Y(a_n)]'$. From the information we are given, we can write the following condition

$$Ab = \mathbf{0},$$

where $\mathbf{0}$ is a $n \times 1$ zero vector. Using the knowledge we learned from linear algebra or Econ 6170, we can show that A is nonsingular. Therefore, the only solution to the linear equation system is

$$b = \mathbf{0}.$$

Then we claim that X and Y follow identical distribution, since $f_X(x) = f_Y(x)$ $\forall x$ on the support of X and Y.