PROF HONG

Probability and Statistics
Chapter # 5

1.(#5.1) A joint PDF is defined by

$$f_{XY}(x,y) = \begin{cases} c(x+2y), & \text{if } 0 < y < 1 \text{ and } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c;

(b) Find the marginal PDF of X;

(c) Find the joint CDF of X and Y;

Solution:

(a) To be a valid PDF, $\int_0^1 \int_0^2 C(x+2y)dxdy = 4C = 1$, therefore $C = \frac{1}{4}$

(b) $f_X(x) = \begin{cases} \int_0^1 \frac{1}{4}(x+2y)dy = \frac{x+1}{4}, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

(c) $F_{XY}(x,y) = \int_0^y \int_0^x \frac{1}{4}(x+2y)dxdy = \frac{1}{8}x^2y + \frac{1}{4}xy^2$,if $0 < x < 2$ and $0 < y < 1$

$F_{XY}(x,y) = \int_0^1 \int_0^x \frac{1}{4}(x+2y)dxdy = \frac{1}{8}x^2 + \frac{1}{4}x$,if $0 < x < 2$ and $y \geq 1$

$F_{XY}(x,y) = \int_0^y \int_0^2 \frac{1}{4}(x+2y)dxdy = \frac{1}{2}y + \frac{1}{2}y^2$,if $x \geq 2$ and $0 < y < 1$

$F_{XY}(x,y) = 1$,if $x \geq 2$ and $y \geq 1$

$F_{XY}(x,y) = 0$,otherwise

2.(#5.2) Suppose (X,Y) has a joint pdf

$$f_{XY}(x,y) = \begin{cases} 1+\theta x & \text{if } -y < x < y, 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where θ is a constant.

(a) Determine the possible value(s) of θ so that $f_{XY}(x,y)$ is a joint PDF. Give your reasoning;

(b) Let $\theta = 0$. Check if X and Y are independent. Give your reasoning.

Solution:

(a) First notice $\int_0^1 \int_{-y}^{y} (1 + \theta x)dxdy = 1$ for any θ. Also we have $f_{XY}(x, y) \geq 0$. From the support, we know $-1 < x < 1$. Suppose $\theta \geq 0$, this implies $1 - \theta < 1 + \theta x < 1 + \theta$. $f_{XY}(x, y) \geq 0$ implies $0 \leq \theta \leq 1$. Similarly when $theta \leq 0$. In conclusion, we have $\theta \in [-1, 1]$.

(b) $f_X(x) = \int_{-x}^{1} 1dy = 1 + x$ if $-1 < x < 0$ and $f_X(x) = \int_x^1 1dy = 1 - x$ if $0 < x < 1$. So $f_X(x) = 1 - |x|$ when $-1 < x < 1$. $f_Y(y) = \int_{-y}^{y} 1dx = 2y$ for $0 < y < 1$. Since $f_{XY}(xy) \neq f_X(x)f_Y(y)$, they are not independent.

3.(#5.4)

(a) Find $P(X > \sqrt{Y})$ if X and Y are jointly distributed with PDF $f_{XY}(x, y) = x + y$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$;

(b) Find $P(X^2 < Y < X)$ if X and Y are jointly distributed with PDF $f_{XY}(x, y) = 2x$ for $0 \leq x \leq 1, 0 \leq y \leq 1$.

Solution:

(a) $P(X > \sqrt{Y}) = \int_0^1 \int_0^{x^2} (x + y)dydx = \frac{7}{20}$

(b) $P(X^2 < Y < X) = \int_0^1 \int_{x^2}^{x} 2xdydx = \frac{1}{6}$

4.(#5.5) Prove that if the jointly CDF of X and Y satisfies $F_{XY}(x, y) = F_X(x)F_Y(y)$, that is, if X and Y are independent, then for any pair of intervals (a, b) and (c, d), $P(a \leq X \leq b, c \leq Y \leq d) = P(a \leq X \leq b)P(c \leq Y \leq d)$.

Solution:
Let $a_{-1} = a$ when X is continuous and and $a_{-1} = argmax_{x\in\Omega_X and x<a}(x - a)$ when X is discrete. Define $c_{-1} = c$ when Y is continuous and and $c_{-1} = argmax_{y\in\Omega_Y and y<c}(y - c)$ when Y is discrete.

$$
\begin{aligned}
P(a &\leq X \leq b, c \leq Y \leq d) = P(X \leq b, c \leq Y \leq d) - P(X \leq a_{-1}, c \leq Y \leq d) \\
&= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c_{-1}) - P(X \leq a_{-1}, Y \leq d) \\
&\quad + P(X \leq a_{-1}, Y \leq c_{-1}) \\
&= F(b, d) - F(b, c_{-1}) - F(a_{-1}, d) + F(a_{-1}, c_{-1}) \\
&= F_X(b)F_Y(d) - F_X(b)F_Y(c_{-1}) - F_X(a_{-1})F_Y(d) \\
&\quad + F_X(a_{-1})F_Y(c_{-1}) \\
&= P(X \leq b)[P(Y \leq d) - P(Y \leq c_{-1})] - P(X \leq a_{-1})[P(Y \leq d) - P(Y \leq c_{-1})] \\
&= P(X \leq b)P(c \leq Y \leq d) - P(X \leq a_{-1})P(c \leq Y \leq d) \\
&= P(a \leq X \leq b)P(c \leq Y \leq d)
\end{aligned}
$$

5.(#5.9) Suppose $g(x) \geq 0$ and $\int_0^\infty g(x)dx = 1$, show that $f(x,y) = \frac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}}$, for $x, y > 0$, is a joint PDF.

Solution:

(i) It is trivial that $f(x,y) \geq 0$. (ii) Substituting x and y with $r\cos\theta$ and $r\sin\theta$ respectively, we obtain

$$\int_0^\infty \int_0^\infty f(x,y)dxdy = \int_0^{\pi/2} \int_0^\infty \frac{2g(r)}{\pi r} r dr d\theta = \frac{2}{\pi} \int_0^{\pi/2} \int_0^\infty g(r) dr d\theta = \frac{2}{\pi} \int_0^{\pi/2} 1 d\theta = 1$$

6. Suppose a joint PDF is

$$f_{XY}(x,y) = \begin{cases} kx, & 0 < x < 1, 0 < y < 1-x, \\ 0, & \text{otherwise.} \end{cases}$$

Find (a) the value of k; (b) the marginal PDF $f_X(x)$; (c) the marginal PDF $f_Y(y)$; (d) the conditional PDF of Y given $X = x$; (e)the conditional PDF of X given $Y = y$; (f)check if X and Y are independent.

Solution:

(a) $\int_0^1 \int_0^{1-x} kxdydx = \frac{k}{6} = 1$, therefore $k = 6$.

(b) $f_X(x) = \begin{cases} \int_0^{1-x} 6xdy = 6x - 6x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f_Y(y) = \begin{cases} \int_0^{1-y} 6xdx = 3(1-y)^2, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$

(d) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-x}, 0 < y < 1-x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(e) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{2x}{(1-y)^2}, & 0 < y < 1-x, 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(f) X and Y are not independent.

7.(#5.10) Suppose (X,Y) has a joint PDF

$$f_{XY}(x,y) = ke^{-y} \text{ for } 0 < x < y < \infty$$

Find (a) the value of k; (b) the marginal PDF $f_X(x)$; (c) the marginal PDF $f_Y(y)$; (d) the conditional PDF of Y given $X = x$; (e)the conditional PDF of X given $Y = y$.

Solution:

(a) $\int_0^\infty \int_0^y ke^{-y}dxdy = \int_0^\infty kye^{-y}dy = k = 1$, therefore $k = 1$.

(b) $f_X(x) = \begin{cases} \int_x^\infty e^{-y}dy = e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

(c) $f_Y(y) = \begin{cases} \int_0^y e^{-y}dx = ye^{-y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$

(d) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} e^{x-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$

(e) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{1}{y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$

8.(#5.11) (X, Y) follows a bivariate normal distribution if their joint PDF

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]},$$

where $-\infty < \mu_1, \mu_2 < \infty, 0 < \sigma_1, \sigma_2 < \infty, -1 \le \rho \le 1$. Find
(a)$f_X(x)$; (b)$f_Y(y)$; (c)$f_{Y|X}(y|x)$; (d)$f_{X|Y}(x|y)$; (e)Under what conditions on parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, X and Y will be independent. [Hint: When finding $f_X(x)$, you can form a term with form

$$z^2 = \left[\left(\frac{y-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2$$

and integrate it out first.]

Solution: [See Chapter 5 section 5.6 pages 149-151.]

9.(#5.6) The random pair (X, Y) has the joint distribution

		X = 1	X = 2	X = 3
Y	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{3}$	0

(a) Show that X and Y are dependent;

(b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

Solution:

(a) To show X and Y are dependent, we can first calculate the marginal pmfs, by summing corresponding rows or columns. We have

$$f_X(x) = \begin{cases} \frac{1}{4}, & x = 1 \\ \frac{1}{2}, & x = 2 \\ \frac{1}{4}, & x = 3 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & x = 2 \\ \frac{1}{3}, & x = 3 \\ \frac{1}{3}, & x = 4 \\ 0, & \text{otherwise} \end{cases}$$

It is easy to see that $f_{XY}(x, y) \neq f_X(x) f_Y(y)$. Thus X and Y are not independent of each other.

(b) The probability table for random variable U and V independent is:

		U		
		1	2	3
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
V	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

10.(#5.7) Suppose X and Y are independent $N(0,1)$ random variables.

(a) Find $P(X^2 + Y^2 < 1)$;

(b) Find $P(X^2 < 1)$, after verifying that X^2 is distributed χ^2_1.

Solution:

(a) By Example 5.21 of the textbook, we know the distribution of $U = X^2 + Y^2$ is $f_U(u) = \frac{1}{2}e^{-\frac{u}{2}}$ when $u > 0$ and 0 otherwise. So $P(U < 1) = \int_0^1 \frac{1}{2}e^{-\frac{u}{2}} du = 1 - e^{-1/2}$.

(b) Using the transformation result, easy to see it follows the pdf of χ^2_1. Then $P(X^2 < 1) = \frac{1}{\pi}(1 - 1/e)ln(\sqrt{2} + 1)$.

11.(#5.8) Let X be an exponential(1) random variable, and define Y to be the integer part of $X + 1$, that is $Y = i + 1$ if and only if $i \leq X < i + 1, i = 0, 1, 2, ...$

(a) Find the distribution of Y. What well-known distribution does Y have?

(b) Find the conditional distribution of $X - 4$ given $Y \geq 5$.

Solution:

(a) Since Y is a discrete random variable, to find the distribution of Y, we just need to find the probability weighting in terms of X for Y taking value y. When $Y = y$, X takes value from $y-1$ to y. Thus $P(Y = y) = P(1 - y \le X \le y) = \int_{y-1}^{y} e^{-x}dx = -e^{-x}|_{y-1}^{y} = e^{-(y-1)} - e^{-y} = (e-1)e^{-y} = (1-\frac{1}{e})e^{1-y} = (1-\frac{1}{e})(\frac{1}{e})^{y-1}$, for $y = 1, 2, 3, ...$, and 0 elsewhere.

Thus Y follows Geometric distribution with $p = 1 - \frac{1}{e}$.

(b) $Y \ge 5 \iff X \ge 4$. The conditional distribution o$fX - 4$ given $Y \ge 5$ is the same with conditional distribution of $X - 4$ given $X \ge 4$. Define $Z = X - 4$, then we try to find $F_{Z|X\ge4}(z|x \ge 4)$.
By definition,

$$\begin{aligned}
F_{Z|X\ge4}(z|x \ge 4) &= P(Z \le z|X \ge 4) \\
&= P(X - 4 \le z|X \ge 4) \\
&= \frac{P(X \le z + 4, X \ge 4)}{P(X \ge 4)} \\
&= \frac{P(4 \le X \le z + 4)}{1 - P(X \le 4)} \\
&= \frac{P(X \le z + 4) - P(X \le 4)}{1 - P(X \le 4)} \\
&= \frac{1 - e^{-(z+4)} - (1 - e^{-4})}{1 - (1 - e^{-4})} \\
&= \frac{e^{-4} - e^{-(z+4)}}{e^{-4}} \\
&= 1 - e^{-z}.
\end{aligned}$$

We can see that the new random variable $Z = X - 4$ conditioning on $X \ge 4$ has an exponential distribution with rate parameter to be 1. Thus, the conditional distribution of $X - 4$ given $X \ge 4$ has identical distribution to X. We call this the memoryless property of exponential distribution.

12.(#5.13) Suppose the random variables X and Y have the following joint pdf

$$f_{XY}(x,y) = \begin{cases} 8xy & \text{for } 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, let $U = X/Y$ and $V = Y$. Determine the joint pdf of U and V.

Solution:
First notice the support is $\Omega_{U,V} = \{(u,v) \in R^2, 0 \le u \le 1, 0 \le v \le 1\}$. And Jacobian $J_{XY}(u,v) = v$. Therefore the joint distribution of $f_{UV}(u,v) = 8uv^3$ on $\Omega_{U,V}$ and 0 elsewhere.

13.(#5.14) (1) Let X_1 and X_2 be independent N(0,1) random variables. Find the PDF of $(X_1 - X_2)^2/2$. (2) If $X_i, i = 1, 2,$are independent Gamma$(\alpha_i, 1)$ random variables, find the marginal distributions of $X_1/(X_1 + X_2)$ and $X_2/(X_1 + X_2)$.

Solution:

(a)
- To solve this question, it is crucial to define the "right" U and V. If we define U to be $\frac{(X_1-X_2)^2}{2}$, then we run into a problem, since no matter how we define V, the mapping is not one-to-one. Thus, let's first focus on the distribution of $\frac{X_1-X_2}{\sqrt{2}}$. The intuition is that X_1 and X_2 are both standard normal random variable and they are independent of each other. Then $X_1 - X_2$ will also be a normal random variable with variance to be 2. Then $\frac{X_1-X_2}{\sqrt{2}}$ should be a standard normal random variable and its square should be a chi-square random variable with degree of freedom 1. With this reasoning, we first construct a bivariate transformation as following:

$$U = \frac{X_1 + X_2}{\sqrt{2}}$$
$$V = \frac{X_1 - X_2}{\sqrt{2}}$$

- The support of (U,V) is $\Omega_{U,V} = \{(u, v) \in R^2, -\infty < u < \infty, -\infty < v < \infty\}$.
- Jacobian $J_{XY}(u, v)$ is computed as following:

$$J_{X,Y}(u, v) = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = 1.$$

- Apply the Bivariate Transformation Theorem, we have the joint distribution of (u,v) is:

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(x, y)\frac{1}{|J_{U,V}(x, y)|} \\ &= \frac{1}{2\pi\sigma^2}exp(-\frac{1}{2\sigma^2}(x^2 + y^2)) \\ &= \frac{1}{2\pi\sigma^2}exp(-\frac{1}{2\sigma^2}(\frac{(u + v)^2}{2} + \frac{(u - v)^2}{2})) \\ &= \frac{1}{2\pi\sigma^2}exp(-\frac{u^2}{2\sigma^2} - \frac{v^2}{2\sigma^2}) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}exp(-\frac{u^2}{2\sigma^2})\frac{1}{\sqrt{2\pi\sigma^2}}exp(-\frac{v^2}{2\sigma^2}) \\ &= f_U(u)f_V(v) \end{aligned}$$

It follows that $U \sim N(0, 1)$ and $V \sim N(0, 1)$. Random variables U and V are independent. $\frac{(X_1-X_2)^2}{2} = V^2 \sim \chi^2(1)$.

(b)
- From $U = \frac{X_1}{X_1+X_2}$ and $V = X_1 + X_2$, we have $X_1 = UV$ and $X_2 = (1 - U)V$.
- The support of (U, V) is $\Omega_{U,V} = \{0 < u < 1, \text{and } 0 < v < \infty\}$.

- The Jacobian $J_{X_1,X_2}(u,v)$ is:

$$J_{X_1,X_2}(u,v) = \begin{vmatrix} v & u \\ -v & (1-u) \end{vmatrix} = v.$$

- Applying the Bivariate Transformation Theorem,

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_1,X_2}(x_1,x_2)|J_{X_1,X_2}(u,v)| \\ &= \frac{1}{\Gamma(\alpha_1)}x_1^{\alpha_1-1}e^{-x_1}\frac{1}{\Gamma(\alpha_2)}x_2^{\alpha_2-1}e^{-x_2}v \\ &= \frac{1}{\Gamma(\alpha_1)}(uv)^{\alpha_1-1}e^{-uv}\frac{1}{\Gamma(\alpha_2)}(1-u)^{\alpha_2-1}v^{\alpha_2-1}e^{-(1-u)v}v \\ &= [\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}u^{\alpha_1-1}(1-u)^{\alpha_2-1}][\frac{1}{\Gamma(\alpha_1+\alpha_2)}v^{\alpha_1+\alpha_2-1}e^{-v}] \end{aligned}$$

  This implies that

$$f_U(u) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)u^{\alpha_1-1}(1-u)^{\alpha_2-1}}$$

  Thus, $U = \frac{X_1}{X_1+X_2} \sim beta(\alpha_1,\alpha_2)$. Similarly, we can show that $\frac{X_2}{X_1+X_2} \sim beta(\alpha_2,\alpha_1)$ from a bivariate transformation when we define $U = \frac{X_2}{X_1+X_2}$ and $V = X_1+X_2$.

14.(#5.15) Suppose X_1, X_2 are independent standard Gamma random variables, possibly with different parameters α_1, α_2. Show:

(a) The random variables

$$X_1 + X_2 \text{ and } \frac{X_1}{X_1+X_2}$$

are mutually independent;

(b) The distribution of $X_1 + X_2$ is a standard Gamma with $\alpha = \alpha_1 + \alpha_2$;

(c) The distribution of $X_1/(X_1+X_2)$ is a standard Beta with parameters α_1, α_2.

Solution:
Similar to part(2) of Question 5.14.

15.(#5.25) Suppose $X_1 \sim$ Gamma$(\alpha_1, 1)$, $X_2 \sim$ Gamma$(\alpha_2, 1)$, and X_1 and X_2 are independent. Show that $X_1 + X_2$ and $X_1/(X_1+X_2)$ are independent. Also, find the marginal distributions of $X_1 + X_2$ and $X_1/(X_1+X_2)$, respectively.

Solution:
From problem 13's result,

$$f_{UV}(u,v) = [\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}u^{\alpha_1-1}(1-u)^{\alpha_2-1}][\frac{1}{\Gamma(\alpha_1+\alpha_2)}v^{\alpha_1+\alpha_2-1}e^{-v}]$$

for $0 < u < 1,$ and $0 < v < \infty.$

It can be written as a produce of function of u and function of v. Thus U and V are independent. It follows that

$$\begin{aligned} f_U(u) &= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1-1}(1-u)^{\alpha_2-1} \\ f_V(v) &= \frac{1}{\Gamma(\alpha_1+\alpha_2)} v^{\alpha_1+\alpha_2-1} e^{-v} \end{aligned}$$

for $0 < u < 1,$ and $0 < v < \infty$, respectively. Thus we know $U = \frac{X_2}{X_1+X_2}$ and $V = X_1 + X_2$ are independent of each other and $U = \frac{X_2}{X_1+X_2}$ follows $beta(\alpha_1, \alpha_2)$ and $V = X_1 + X_2$ follows $Gamma(\alpha_1 + \alpha_2, 1)$.

16.(#5.16) X_1 and X_2 are independent $N(0, \sigma^2)$ random variables.

(1) Find the joint distribution of Y_1 and Y_2, where $Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_1/\sqrt{Y_1}$.

(2) Show that Y_1 and Y_2 are independent.

Solution:

You can read Chapter 5 Example 21 to solve this question. When you solve this one, you have to change your notation.

(a)

- Find we need to find out the support for Y_1 and Y_2: from $Y_1 = X_1^2 + X_2^2$, and $Y_2 = \frac{X_1}{\sqrt{X_1^2+X_2^2}}$, and $X_1 \sim N(0, \sigma^2)$, $X_2 \sim N(0, \sigma^2)$, we can derive the support for (Y_1, Y_2) as:

$$\Omega_{Y_1,Y_2} = \{(y_1, y_2) \in R^2 : 0 < y_1 < \infty, -1 < y_2 < 1\}.$$

- Be careful, **Bivariate Transformation Theorem doesn't apply here!**, since (Y_1, Y_2) is not a 1-1 mapping from (X_1, X_2). For the distribution of $Z = X_2^2$, we have for the CDF of Z, $\forall z \in \Omega_Z = \{z \in [0, \infty)\}$:

$$\begin{aligned} F_Z(z) &= P(X_2^2 \le z) \\ &= P(-\sqrt{z} \le X_2 \le \sqrt{z}) \\ &= F_{X_2}(\sqrt{z}) - F_{X_2}(-\sqrt{z}) \end{aligned}$$

The pdf of Z is:

$$\begin{aligned} f_Z(z) &= \frac{dF_Z(z)}{dz} \\ &= f_{X_2}(\sqrt{z})\frac{1}{2\sqrt{z}} + f_{X_2}(-\sqrt{z})\frac{1}{2\sqrt{z}} \\ &= \frac{1}{\sqrt{2\pi z\sigma^2}} e^{-\frac{z}{2\sigma^2}}, \text{for } z \in [0, \infty) \end{aligned}$$

and 0, elsewhere.

- X_1 is independent of X_2, then X_2 is also independent of $Z \equiv X_2^2$. The joint pdf of (X_1, Z) is:

$$f_{X_1,Z}(x_1, z) = \frac{1}{2\pi\sigma^2\sqrt{z}} e^{-x^2/2\sigma^2} e^{-z/2\sigma^2}$$

The support of (X_1, Z) is $\Omega_{X_1,Z} = \{(x_1, z) \in R^2, -\infty < x_1 < -\infty, 0 \leq z < \infty\}$. After substituting X_2^2 by Z, we can do the following transformation:

$$Y_1 = X_1^2 + Z$$
$$Y_2 = \frac{X_1}{\sqrt{X_1^2 + Z}}$$

then,

$$X_1 = \sqrt{Y_1}Y_2$$
$$Z = Y_1(1 - Y_2^2)$$

- The Jacobian $J_{X_1,Z}(y_1, y_2)$ is:

$$J_{X_1,Z}(y_1, y_2) = \begin{vmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ 1 - y_2^2 & -2y_1y_2 \end{vmatrix} = -\sqrt{y_1}.$$

- Using Bivariate Transformation Theorem, we have:

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,Z}(x_1, z)|J_{X_1,Z}(y_1, y_2)| \\ &= \frac{1}{2\pi\sigma^2\sqrt{1 - y_2^2}} e^{-\frac{y_1}{2\sigma^2}}, \quad \text{for } (y_1, y_2) \in \Omega_{Y_1Y_2} \end{aligned}$$

and 0 otherwise.

(b) Y_1 and Y_2 are independent, because, the joint distribution equals the product of the marginal distribution for all $(y_1, y_2) \in \Omega_{Y_1Y_2}$. First calculate the marginal pdf of Y_1:

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-1}^{1} \frac{1}{2\pi\sigma^2\sqrt{1 - y_2^2}} e^{-\frac{y_1}{2\sigma^2}} dy_2 \\ &= \frac{1}{2\sigma^2} e^{-\frac{y_1}{2\sigma^2}} \int_{-1}^{1} \frac{1}{\pi\sqrt{1 - y_2^2}} dy_2 \\ &= \frac{1}{2\sigma^2} e^{-\frac{y_1}{2\sigma^2}} \int_{\pi}^{0} \frac{1}{\pi\sqrt{1 - cos(\theta)^2}} dcos(\theta) \\ &= \frac{1}{2\sigma^2} e^{-\frac{y_1}{2\sigma^2}} \int_{\pi}^{0} \frac{1}{\pi sin(\theta)} (-sin(\theta)) d\theta \\ &= \frac{1}{2\sigma^2} e^{-\frac{y_1}{2\sigma^2}} (-\frac{1}{\pi})\theta|_{\pi}^{0} \\ &= \frac{1}{2\sigma^2} e^{-\frac{y_1}{2\sigma^2}}, \text{for } y_1 \in (0, \infty), \end{aligned}$$

and 0, elsewhere. Actually, we can see that Y_1 follows Exponential($\frac{1}{2\sigma^2}$). By the same logic, we can show that

$$f_{Y_2}(y_2) = \frac{1}{\pi\sqrt{1-y_2^2}}, \text{for } y_2 \in (-1, 1)$$

and 0, elsewhere. Then we can prove Y_1 and Y_2 are independent of each other since

$$f_{Y_1Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2).$$

You can also prove independence by factorization theorem like the lecture note does, however, you have to show that the result holds for all (y_1, y_2) pair in the R^2 space.

17.(#5.17) For $X \sim Beta(\alpha, \beta)$, and $Y \sim Beta(\alpha+\beta, \gamma)$ be independent random variables, find the distribution of XY by making the transformation given in (1) and (2) and integrating out V.

(a) $U = XY, V = Y$.

(b) $U = XY, V = X/Y$.

Solution:

(a) From $U = XY$ and $V = Y$, we have $X = \frac{U}{V}$ and $Y = V$. We can first find the support is $0 < u < v < 1$.
The determinant of Jacobian matrix is:

$$|\det J_{XY}(u, v)| = \left|\det \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{bmatrix}\right| = \frac{1}{v}$$

It follows that

$$\begin{aligned} f_{UV}(u, v) &= f_{XY}(x, y)|J_{XY}(u, v)| = f_{XY}(x, y)\frac{1}{v} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)}y^{\alpha+\beta-1}(1-y)^{\gamma-1}\frac{1}{v} \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}(\frac{u}{v})^{\alpha-1}(1-\frac{u}{v})^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}\frac{1}{v}, \ 0 < u < v < 1 \end{aligned}$$

Then,

$$\begin{aligned} f_U(u) &= \int_u^1 f_{UV}(u, v)dv \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}\int_u^1 v^{\beta-1}(\frac{v-u}{v})^{\beta-1}(1-v)^{\gamma-1}dv \end{aligned}$$

Let $z = \frac{v-u}{1-u}$, we have $dz = \frac{dv}{1-u}$, $1 - v = (1-y)(1-u)$ and $\frac{v-u}{v} = \frac{z(1-u)}{v}$. Thus

$$\begin{aligned}
f_U(u) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_0^1 v^{\beta-1}(1-z)^{\gamma-1}(1-u)^{\gamma-1} \frac{z^{\beta-1}(1-u)^{\beta-1}}{v^{\beta-1}}(1-u)dz \\
&= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_0^1 (1-z)^{\gamma-1} z^{\beta-1} dz \\
&= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\
&= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}, \quad 0 < u < 1
\end{aligned}$$

Thus, $U \sim beta(\alpha, \beta+\gamma)$.

(b) From $U = XY$ and $V = X/Y$, we have $X = \sqrt{UV}$ and $Y = \sqrt{\frac{U}{V}}$. The support of (U, V) is $0 < u < v < 1/u$ and $0 < u < 1$.
The determinant of Jacobian matrix is:

$$|\det J_{XY}(u,v)| = \left| \det \begin{bmatrix} \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \\ \frac{1}{2}\sqrt{\frac{1}{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \end{bmatrix} \right| = \frac{1}{2v}$$

It follows that

$$\begin{aligned}
f_{UV}(u,v) &= f_{XY}(x,y)|J_{XY}(u,v)| = f_{XY}(x,y)\frac{1}{2v} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1}\frac{1}{2v} \\
&= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sqrt{uv}^{\alpha-1}(1-\sqrt{uv})^{\beta-1} \sqrt{\frac{u}{v}}^{\alpha+\beta-1} (1-\sqrt{\frac{u}{v}})^{\gamma-1}\frac{1}{2v}
\end{aligned}$$

The set $\{0 < x < 1, 0 < y < 1\}$ is mapped onto the set $\{0 < u < v < \frac{1}{u}, 0 < u < 1\}$. Then,

$$\begin{aligned}
f_U(u) &= \int_u^{1/u} f_{UV}(u,v)dv \\
&= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_u^{1/u} (\frac{1-\sqrt{uv}}{1-u})^{\beta-1}(\frac{1-\sqrt{\frac{u}{v}}}{1-u})^{\gamma-1} \\
&\quad \times \frac{\sqrt{\frac{u}{v}}^{\beta}}{2v(1-u)} dv \\
&= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_u^{1/u} (\frac{\sqrt{\frac{u}{v}}-u}{1-u})^{\beta-1}(1-\frac{\sqrt{\frac{u}{v}}-u}{1-u})^{\gamma-1} \\
&\quad \times \frac{\sqrt{\frac{u}{v}}}{2v(1-u)} dv
\end{aligned}$$

Let $z = \frac{\sqrt{\frac{u}{v}} - u}{1-u}$, we have $dz = -\frac{\sqrt{\frac{u}{v}}}{2(1-u)v} dv$. Thus

$$\begin{aligned} f_U(u) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_0^1 z^{\beta-1}(1-z)^{\gamma-1} dz \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}, \quad 0 < u < 1 \end{aligned}$$

Thus, $U \sim beta(\alpha, \beta+\gamma)$.

18.(#5.18) Let $X \sim N(\mu, \sigma^2)$, and let $Y \sim N(\gamma, \sigma^2)$. Suppose X and Y are independent. Define $U = X + Y$ and $V = X - Y$. Show that U and V are independent normal random variables. Find the distribution of each of them.

Solution:
Let $Z = Y + (\mu - \gamma)$. Then X and Z are independent and $Z \sim N(\mu, \sigma^2)$. From the result of Chapter 5 example 22, $X + Z \sim N(2\mu, 2\sigma^2), X - Z \sim N(0, 2\sigma^2)$, and $X + Z, X - Z$ are independent. It implies that $U = X + Z - (\mu - \gamma) \sim N(\mu + \gamma, 2\sigma^2)$, $V = X - Z + (\mu - \gamma) \sim N(\mu - \gamma, 2\sigma^2)$ are also independent, since adding constant does not change independence.

19.(#5.20) Suppose $X_1 \sim N(0,1), X_2 \sim N(0,1)$ and X_1 and X_2 are independent. Find the distribution of X_1/X_2.

Solution:
From $U = X_1/X_2$ and $V = X_2$, we have $X_1 = UV$ and $X_2 = V$. The support for (U, V) is $-\infty < u < \infty$ and $-\infty < v < \infty$. The determinant of Jacobian matrix is:

$$|\det J_{X_1X_2}(u,v)| = \left| \det \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} \right| = |v|$$

It follows that

$$\begin{aligned} f_{UV}(u,v) &= f_{X_1X_2}(x_1,x_2)|J_{X_1X_2}(u,v)| = f_{X_1X_2}(x_1,x_2)|v| \\ &= \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} |v| \\ &= \frac{|v|}{2\pi} e^{-(u^2+1)v^2/2} \end{aligned}$$

Then,

$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{UV}(u,v)dv \\
&= \int_{-\infty}^{\infty} \frac{|v|}{2\pi} e^{-(u^2+1)v^2/2} dv \\
&= \int_{-\infty}^{0} \frac{-v}{2\pi} e^{-(u^2+1)v^2/2} dv + \int_{0}^{\infty} \frac{v}{2\pi} e^{-(u^2+1)v^2/2} dv \\
&= 2\int_{0}^{\infty} \frac{v}{2\pi} e^{-(u^2+1)v^2/2} dv \\
&= \frac{1}{\pi}\int_{0}^{\infty} v e^{-(u^2+1)v^2/2} dv
\end{aligned}
$$

let $z = (u^2+1)v^2/2$, then we have $\frac{dz}{dv} = v(u^2+1)$, thus we have $dv = \frac{dz}{v(u^2+1)}$. Also, we have $e^{-(u^2+1)v^2/2} = e^{-z}$. Substituting them back to $f_U(u)$, we have

$$
\begin{aligned}
f_U(u) &= \frac{1}{\pi}\int_{0}^{\infty} v e^{-z} \frac{dz}{v(u^2+1)} \\
&= \frac{1}{\pi(u^2+1)} \int_{0}^{\infty} e^{-z} dz \\
&= \frac{1}{\pi(u^2+1)} [-e^{z}|_0^{\infty}] \\
&= \frac{1}{\pi(u^2+1)} \text{ for } u \in (-\infty, \infty)
\end{aligned}
$$

Thus, $U \sim Cauchy(0,1)$.

20.(#5.21) Let Z_1, Z_2 be independent standard normal random variables. Define

$$
\begin{aligned}
X &= \mu_1 + aZ_1 + bZ_2, \\
Y &= \mu_2 + cZ_1 + dZ_2,
\end{aligned}
$$

where constants a, b, c, d satisfy the restrictions that

$$
\begin{aligned}
a^2 + b^2 &= \sigma_1^2, \\
c^2 + d^2 &= \sigma_2^2, \\
ac + bd &= \rho\sigma_1\sigma_2.
\end{aligned}
$$

Show that $(X,Y) \sim BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

Solution:
Representing Z_1 and Z_2 using X and Y:

$$
\begin{aligned}
Z_1 &= \frac{d\mu_1 - b\mu_2 + bY - dX}{bc - ad} \\
Z_2 &= \frac{a\mu_2 - c\mu_1 + cX - aY}{bc - ad}
\end{aligned}
$$

Therefore $|detJ_{Z_1Z_2}(x,y)| = \frac{|ad-bc|}{(bc-ad)^2} = \frac{1}{|bc-ad|} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}$

$$\begin{aligned}
f_{XY}(x,y) &= f_{Z_1Z_2}(z_1,z_2)|detJ_{Z_1Z_2}(x,y)| \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z_2^2}{2}}\frac{1}{|bc-ad|} \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2}\frac{(d\mu_1-b\mu_2)^2+b^2Y^2+d^2X^2-bdXY+(a\mu_2-c\mu_1)^2+cX^2+aY^2-acX}{(bc-ad)^2}} \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\left(\frac{y-\mu_2}{\sigma_2}\right)^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]}
\end{aligned}$$

21.(#5.22) Let X and Y be two independent uniform random variables on $[0,1]$. Show that the random variables $U = \cos(2\pi X)\sqrt{-2\ln Y}$ and $V = \sin(2\pi X)\sqrt{-2\ln Y}$ are independent standard normal random variables.

Solution:

This is the so called Box-Muller Transformation. Expressing X and Y in terms of U and V:

$$\begin{aligned}
X &= \frac{1}{2\pi}tan^{-1}\left(\frac{V}{U}\right) \\
Y &= e^{-(U^2+V^2)/2}
\end{aligned}$$

Then easy to see $|detJ_{XY}(u,v)| = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}$.

22.(#5.23) Find the PDF of $X - Y$, where $X \sim U[0,1], Y \sim U[0,1]$, and X and Y are independent.

Solution:

See Example 5.23 in section 5.5.

23.(#5.65) Suppose X has a probability density function

$$f(x) = \begin{cases} |x| & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = X^2$.

(a) Find $Cov(X,Y)$; (b) Are X and Y independent? Give your reasoning.

Solution:

(a) Using the formula for covariance function, we have $Cov(X,Y) = EXY - EXEY = EX^3 - EXEY$. Here we can see that the density of X is symmetric about $y-axis$, then we expect the mean and third moment of X are 0. To see this, let k be an odd number, we calculate $E(X^k)$.

$$\begin{aligned} E(X^k) &= \int_{-1}^{1} x^k|x|dx \\ &= \int_{-1}^{0} -x^{k+1}dx + \int_{0}^{1} x^{k+1}dx \\ &= -\frac{x^{k+2}}{k+2}|_{-1}^{0} + \frac{x^{k+2}}{k+2}|_{0}^{1} \\ &= \frac{-1}{k+2} + \frac{1}{k+2} \\ &= 0 \end{aligned}$$

Thus, we know that all the odd moments of X are 0. Then we have $cov(X,Y) = 0$.

(b) Are X and Y independent? No, because it is given that Y is a function of X. X and Y are not independent even though they are uncorrelated.

24.(#5.44). Suppose (X,Y) has a bivariate normal PDF

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}.$$

Show that $\text{corr}(X,Y) = \rho$.

Solution:

We first compute the marginal density of X and Y separately. The marginal density of X, is:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho)^2}} e^{-\frac{y^2-2\rho xy+(\rho x)^2+x^2-(\rho x)^2}{2(1-\rho^2)}} dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times \frac{1}{\sqrt{2\pi(1-\rho^2)}} \int_{-\infty}^{\infty} e^{(y-\rho x)^2/2(1-\rho^2)} dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \end{aligned}$$

Thus, X is $N(0,1)$. Similarly, $Y \sim N(0,1)$. $EX = EY = 0$.
$corr(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{EXY-0}{1} = EXY$. Then, we compute the covariance between X and Y.

$$\begin{aligned} EXY &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \frac{1}{2\pi\sqrt{(1-\rho^2)}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}} dxdy \\ &= \frac{\rho}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \\ &= \rho, \end{aligned}$$

where the second to las equality comes from that the the mean for a random variable Y that follows $N(\rho x, 1-\rho^2)$ is ρx, and the last equality is because the second moment of a standard normal random variable is 1.

25. A normal distributed, denoted as $N\left(\mu, \sigma^2\right)$, random variable has the moment generating function $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ for $-\infty < t < \infty$. Suppose $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$, and X and Y are independent. Find the distribution of $a_1 X + a_2 Y$. Give your reasoning.

Solution:

Denote $Z = a_1 X + a_2 Y$, then we need to find the distribution for Z. By the uniqueness theorem of MGF, we can identify the distribution of Z from its MGF.

$$\begin{aligned}
M_Z(t) &= E\left(e^{tZ}\right) \\
&= E\left(e^{t(a_1 X + a_2 Y)}\right) \\
&= E\left(e^{a_1 t X} e^{a_2 t Y}\right) \\
\text{by independence } &= E\left(e^{a_1 t X}\right) E\left(e^{a_2 t Y}\right) \\
\text{by definition of MGF} &= M_X\left(a_1 t\right) M_Y\left(a_2 t\right) \\
&= e^{\mu_1 a_1 t + \frac{1}{2}\sigma_1^2 a_1^2 t^2} e^{\mu_2 a_2 t + \frac{1}{2}\sigma_2^2 a_2^2 t^2} \\
&= e^{(\mu_1 a_1 + \mu_2 a_2)t + \frac{1}{2}\left(\sigma_1^2 a_1^2 + \sigma_2^2 a_2^2\right)t^2} \\
&= e^{\mu t + \frac{1}{2}\sigma^2 t^2}
\end{aligned}$$

where $\mu = \mu_1 a_1 + \mu_2 a_2$ and $\sigma^2 = \sigma_1^2 a_1^2 + \sigma_2^2 a_2^2$. Then we know that $Z = a_1 X + a_2 Y$ follows normal distribution with mean $\mu_1 a_1 + \mu_2 a_2$ and variance $\sigma_1^2 a_1^2 + \sigma_2^2 a_2^2$.

26. (# 5.47). Suppose the joint PDF of X, Y is a uniform PDF on the circle $x^2 + y^2 \leq 1$. Find (1) $E(Y|X)$; (2) var$(Y|X)$; (3) Are X and Y independent? Explain.

Solution:

(a) $f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi}$ for $-1 \leq x \leq 1$. Then we easily get $f_{Y|X}(y|x) = \frac{1}{2\sqrt{1-x^2}}$. Therefore

$$\begin{aligned}
E(Y|X) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \frac{1}{2\sqrt{1-x^2}} dy \\
&= \frac{1}{4\sqrt{1-x^2}} (y^2|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}) = 0
\end{aligned}$$

(b) Similarly we can calculate $E(Y^2|X) = \frac{1-x^2}{3}$. Then

$$\begin{aligned}
var(Y|X) &= E(Y^2) - (E(Y|X))^2 \\
&= \frac{1-x^2}{3}
\end{aligned}$$

(c) Since the conditional covariance is not a constant, X and Y are not independent.

27. Suppose (X, Y) have a joint PDF

$$f_{XY}(x,y) = \begin{cases} e^{-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $E(Y|X)$. Can you use X to predict $E(Y|X)$? Explain.

(b) Find $Var(Y|X)$. Can you use X to predict $Var(Y|X)$? Explain.

Solution:

Since we are dealing with the conditional mean and conditional variance on X, we need first calculate the marginal distribution of X.

$$\begin{aligned} f_X(x) &= \int_x^\infty e^{-y} dy \\ &= -\left. e^{-y}\right|_0^\infty \\ &= e^{-x}, \text{ for } x \in (0, \infty), \end{aligned}$$

and 0 elsewhere. Next, we calculate the conditional pdf $f_{Y|X}(y \mid x)$:

$$\begin{aligned} f_{Y|X}(y \mid x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= e^{x-y}, \text{ for } y \in (x, \infty), x \in (0, \infty) \end{aligned}$$

and 0 elsewhere.

(a) By definition

$$\begin{aligned} E(Y \mid X) &= \int_x^\infty y e^{x-y} dy \\ &= e^x \int_x^\infty y e^{-y} dy \\ &= e^x \int_x^\infty (-y) d e^{-y} \\ &= e^x \left[(-y) e^{-y}\Big|_0^\infty + \int_x^\infty e^{-y} dy \right] \\ &= e^x \left(x e^{-x} + e^{-x} \right) \\ &= x + 1, \text{ for } x \in (0, \infty) \end{aligned}$$

Since the conditional mean function is a function of X, then we can use X to predict $E(Y \mid X)$.

(b) To find $\operatorname{Var}(Y \mid X)$, we need to first find $E\left(Y^2 \mid X\right)$.

$$\begin{aligned}
E\left(Y^2 \mid X\right) &= \int_x^\infty y^2 e^{x-y} dy \\
&= e^x \int_x^\infty y^2 e^{-y} dy \\
&= e^x \int_x^\infty \left(-y^2\right) de^{-y} \\
&= e^x \left[\left(-y^2\right) e^{-y}\Big|_0^\infty + 2\int_x^\infty y e^{-y} dy\right] \\
&= e^x \left(x^2 e^{-x} + 2e^{-x} x + 2e^{-x}\right) \\
&= x^2 + 2x + 2, \text{ for } x \in (0, \infty)
\end{aligned}$$

Then we know $\operatorname{Var}(Y \mid X) = E\left(Y^2 \mid X\right) - [E(Y \mid X)]^2 = 1$. Since the conditional variance is a constant, we cannot use X to predict $\operatorname{Var}(Y \mid X)$

28. Show that the conditional mean $E(Y|X)$ is the optimal minimizer for the minimization problem of the mean squared error $E[Y - g(X)]^2$;that is

$$E(Y|X) = arg\min E[Y - g(X)]^2,$$

where the minimization is over all measurable and square-integrable functions.

Solution: See Theorem 5.25 in textbook.

29. Let X and Y be two random variables and $0 < \sigma_X^2 < \infty$. Show that if $E(Y|X) = a + bX$, then $b = cov(X, Y)/\sigma_X^2$.

Solution: We know $\operatorname{Cov}(X, Y) = E(XY) - EXEY$. By Law of Iterated Expectation, we have $\operatorname{Cov}(X, Y) = E[XE(Y \mid X)] - \mathrm{E}\,X\,\mathrm{E}[E(Y \mid X)]$. Give $E(Y \mid X) = a + bX$, we have

$$\begin{aligned}
\operatorname{Cov}(X, Y) &= E[X(a + bX)] - EXE(a + bX) \\
&= E\left[aX + bX^2\right] - EX(a + bEX) \\
&= aEX + bEX^2 - aEX - b(EX)^2 \\
&= b\left[EX^2 - (EX)^2\right] \\
&= b\sigma_X^2
\end{aligned}$$

Given $0 < \sigma_X^2 < \infty$, dividing σ_X^2 on both sides will give us $b = \operatorname{Cov}(X, Y)/\sigma_X^2$.

30. Suppose $E(Y|X) = 1 + 2X$ and $var(X) = 2$. Find $cov(X, Y)$.

Solution: From the result in Q **22 note to change the order**, we know $\operatorname{Cov}(X, Y) = b * \operatorname{Var}(X)$. In this question, $\operatorname{Var}(X) = 2$ and $b = 2$. Then we have $\operatorname{Cov}(X, Y) = 2 * 2 = 4$.

31.(# 5.50) Suppose X and Y are random vaiables such that $E\left(E(Y|X\right) = 7 - \frac{1}{4}X$ and $E\left(E(X|Y)\right) = 10 - Y$. Determine the correlation between X and Y.

Solution:

$$\begin{aligned} E(Y) &= E(E(Y|X)) = 7 - \frac{1}{4}E(X) \\ E(X) &= E(E(X|Y)) = 10 - E(Y) \end{aligned}$$

We get $E(X) = 4$ and $E(Y) = 6$.

$$\begin{aligned} E(XY) &= E(E(XY|X)) = E(XE(Y|X)) \\ &= 7E(X) - \frac{1}{4}E(X^2) \\ &= 28 - \frac{1}{4}(Var(X) + E^2(X)) \\ &= 24 - \frac{1}{4}Var(X) \\ &\Longrightarrow Var(X) = 4(24 - E(XY)) \end{aligned}$$

$$\begin{aligned} E(XY) &= E(E(XY|Y)) = E(YE(X|Y)) \\ &= 10E(Y) - E(Y^2) \\ &= 60 - (Var(Y) + E^2(Y)) \\ &= 24 - Var(Y) \\ &\Longrightarrow Var(Y) = 24 - E(XY) \end{aligned}$$

$$\begin{aligned} Corr(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{Var(X)Var(Y)}} \\ &= \frac{E(XY) - 24}{\sqrt{(24 - E(XY))4(24 - E(XY))}} = -\frac{1}{2}. \end{aligned}$$

32. Show $\text{var}(Y|X) = E\left(Y^2|X\right) - [E(Y|X)]^2$.

Solution:

By defintion

$$\begin{aligned}
\operatorname{Var}(Y \mid X=x) &= \int [y - E(Y \mid X=x)]^2 \, dF_{Y|X}(y \mid x) \\
&= \int \left[y^2 - 2yE(Y \mid X=x) + E(Y \mid X=x)^2\right] dF_{Y|X}(y \mid x) \\
&= \int y^2 dF_{Y|X}(y \mid x) - 2E(Y \mid X=x) \int y dF_{Y|X}(y \mid x) \\
&\quad + E(Y \mid X=x)^2 \int dF_{Y|X}(y \mid x) \\
&= E\left(Y^2 \mid X=x\right) - 2E(Y \mid X=x)E(Y \mid X=x) + E(Y \mid X=x)^2 \\
&= E\left(Y^2 \mid X=x\right) - E(Y \mid X=x)^2
\end{aligned}$$

33.(# 5.60) For any two random variables X and Y with finite variances, show:

(a) $cov(X,Y) = cov\,(X, E(Y|X))$.

(b) X and $Y - E(Y|X)$ are uncorrelated.

(c) $var\,[Y - E(Y|X)] = E\,[var(Y|X)]$.

Solution:

(a)

$$\begin{aligned}
Cov(X, E(Y|X)) &= E(XE(Y|X)) - E(X)E(E(Y|X)) = E(E(XY|X)) - E(X)E(Y) \\
&= E(XY) - E(X)E(Y) = Cov(X,Y)
\end{aligned}$$

(b)

$$\begin{aligned}
Cov(X, Y - E(Y|X)) &= E(X(Y - E(Y|X))) - E(X)E(Y - E(Y|X)) \\
&= E(XY - XE(Y|X)) - E(X)(E(Y) - E(E(Y|X))) \\
&= E(XY) - E(E(XY|X)) - E(X)(E(Y) - E(Y)) \\
&= E(XY) - E(XY) = 0
\end{aligned}$$

(c)

$$\begin{aligned}
Var(Y - E(Y|X)) &= E(Y - E(Y|X))^2 - E^2(Y - E(Y|X)) \\
&= E(Y - E(Y|X))^2 \\
&= E(Y^2 - 2YE(Y|X) + E^2(Y|X)) \\
&= E(E(Y^2|X) - 2E(Y|X)E(Y|X) + E^2(Y|X)) \\
&= E(E(Y^2|X) - E^2(Y|X)) = E(Var(Y|X))
\end{aligned}$$

34.(# 5.61) (a)Suppose $E(Y|X) = E(Y)$. Show $cov(X,Y) = 0$. (b) Does $cov(X,Y) = 0$ imply $E(Y|X) = E(Y)$? If yes, prove it. If not, provide an example.

Solution:

(a)

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(E(XY|X)) - E(X)E(Y) \\ &= E(XE(Y|X)) - E(X)E(Y) \\ &= E(XE(Y)) - E(X)E(Y) \\ &= E(X)E(Y) - E(X)E(Y) \\ &= 0 \end{aligned}$$

(b) No. $E(Y|X) = 1(|X - \theta| < 2)$ varies with X.

35.(#5.32) Suppose the distribution of Y, conditional on $X = x$, is $N(x, x^2)$ and that the marginal distribution of X is uniform(0,1).

(a) Find $E(Y)$, var(Y), and cov(X,Y);

(b) Prove that Y/X and X are independent.

Solution:

(a) $E(Y) = E(E(Y|X)) = E(X) = 1/2$
$E(Y^2) = E(E(Y^2|X)) = 2E(X^2) = 2/3$
$var(Y) = E(Y^2) - E(Y)^2 = 5/12$
$cov(X,Y) = E(XY) - E(X)E(Y) = E(X(E(Y|X))) - 1/4 = 1/12$

(b) Let $U = Y/X$ and $V = X$, easy to show that $f_{UV}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-1)^2}{2}}$. By factorization theorem, U and V are independent.

36.(#5.33) Consider two random variables (X,Y). Suppose X is uniformly distributed over $(-1,1)$, that is, the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, the conditional pdf of Y give $X = x$ is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\alpha-\beta x)^2}{2}} \quad \text{for } -\infty < y < \infty \text{ and } -1 < x < 1 \ .$$

Find: (a) $E(Y)$; (b) $\text{cov}(X, Y)$.

Solution:

(a) Since the conditional pdf of Y given X is a normal distribution, we know $E(Y|X) = \alpha + \beta X$. Then $E(Y) = E(E(Y|X)) = \alpha$

(b) $cov(X, Y) = E(XY) - E(X)E(Y) = E(\alpha X + \beta X^2) = \beta/3$

37.(#5.38) A generalization of the beta distribution is the *Dirichlet* distribution. In its bivariate version, (X, Y) have a joint PDF $f_{XY}(x, y) = kx^{a-1}y^{b-1}(1 - x - y)^{c-1}$, $0 < x < 1$, $0 < y < 1$, $0 < y < 1 - x < 1$, where $a > 0$, $b > 0$, and $c > 0$ are constants.

(a) Show that $k = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$;

(b) Show that, marginally, both X and Y are Beta;

(c) Find the conditional distribution of $Y|X = x$, and show that $Y|(1 - X)$ is Beta(b, c);

(d) Show that $E(XY) = \frac{ab}{(a+b+c+1)(a+b+c)}$, and find the covariance $\text{cov}(X, Y)$.

Solution:

(a) Let $z = \frac{y}{1-x}$, we have

$$\begin{aligned}
\int\int f_{XY}(x, y)dxdy &= 1 \\
\int_0^1 \int_0^{1-x} kx^{\alpha-1}y^{b-1}(1 - x - y)^{c-1}dxdy &= 1 \\
\int_0^1 \int_0^1 kx^{a-1}z^{b-1}(1 - x)^{b-1}(y/z - y)^{c-1}(1 - x)dzdx &= 1 \\
kB(a, b + c)B(b, c) &= 1
\end{aligned}$$

(b) Similar to part (a), we still let $z = \frac{y}{1-x}$:

$$\begin{aligned}
f_X(x) &= \int_0^1 kx^{a-1}(1 - x)^{b+c+1}z^{b-1}(1 - z)^{c-1}dz \\
&= \frac{\Gamma(a + b + c)}{\Gamma(a)\Gamma(b + c)}x^{a-1}(1 - x)^{b+c-1}
\end{aligned}$$

$f_Y(y)$ is similar.

(c)

$$\frac{f_{XY}(x, y)}{f_X(x)} = \frac{\Gamma(b + c)}{\Gamma(b)\Gamma(c)} \frac{y^{b-1}(1 - x - y)^{c-1}}{(1 - x)^{b+c-1}}$$

for $0 < x < 1, 0 < y < 1, 0 < y < 1 - x < 1$. Let $U = Y/(1-X)$ and $V = 1 - X$. Easy to show

$$f_{UV}(u,v) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} v^{b+c-1}(1-v)^{a-1} \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} u^{b-1}(1-u)^{c-1}$$

Then

$$\begin{aligned} f_U(u) &= \int_0^1 \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} v^{b+c-1}(1-v)^{a-1} \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} u^{b-1}(1-u)^{c-1} dv \\ &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} u^{b-1}(1-u)^{c-1} \int_0^1 \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} v^{b+c-1}(1-v)^{a-1} dv \\ &= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} u^{b-1}(1-u)^{c-1} \end{aligned}$$

for $0 < u < 1$

(d)

$$\begin{aligned} E(XY) &= E(X(1-X)E(\frac{Y}{1-X}|X)) \\ &= E(X(1-X)\frac{b}{b+c}) \\ &= \frac{b}{b+c}(\frac{a}{a+b+c} - \frac{a}{a+b+c}\frac{a+1}{a+b+c+1}) \\ &= RHS \end{aligned}$$

$cov(X,Y) = E(XY) - E(X)E(Y) = RHS$

38.(#5.45) Suppose (X,Y) follows a standard bivariate normal distribution with correlation coefficient ρ. Define $U = (Y - \rho X)/\sqrt{1-\rho^2}$. Show that U is normally distributed and independent of X.

Solution: Let $V = X$. We have

$$\begin{aligned} f_{UV}(u,v) &= f_{XY}(x,y)|det(J_{UV}(x,y))|^{-1} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)} \sqrt{1-\rho^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \end{aligned}$$

By factorization theorem, U and X are independent. Also, $f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$.

39.(#5.57) Suppose (X,Y) have a joint PDF

$$f_{XY}(x,y) = \begin{cases} xe^{-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the conditional pdf $f_{Y|X}(y|x)$ of Y given $X = x$.

(b) Find the conditional mean $E(Y|x)$;

(c) Find the conditional variance $\text{var}(Y|x)$;

(d) Are X and Y independent? Give your reasoning.

Solution:

(a) First get $f_X(x) = xe^{-x}$ for $0 < x < \infty$. Then $f_{Y|X}(y|x) = \frac{xe^{-y}}{xe^{-x}} = e^{x-y}$ for $0 < x < y < \infty$.

(b)

$$
\begin{aligned}
E(Y|X) &= \int_x^{\infty} y e^{x-y} dy \\
&= x + 1
\end{aligned}
$$

(c) Similarly we get $E(Y^2|X) = x^2+2x+2$. Then we have $var(Y|X) = E(Y^2)-(E(Y|X))^2 = 1$

(d) $f_Y(y) = \frac{1}{2}y^2 e^{-y}$ for $0 < y < \infty$. Since $f_{XY}(x, y) \neq f_X(x)f_Y(y)$, they are not independent.