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Probability and Statistics for Economists
Chapter 2 Foundation of Probability Theory

1. For event A and B, find formula for the probabilities of the following events in terms of the quantities $P(A)$, $P(B)$, and $P(A \cap B)$:
(1) either A or B or both;
(2) either A or B but not both;
(3) at least one of A or B;
(4) at most one of A or B.

Solution:
(1) "A or B or both" is $A \cup B$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(2) "A or B but not both" is $(A \cap B^c) \cup (B \cap A^c)$

$$\begin{aligned} P((A \cap B^c) \cup (B \cap A^c)) &= P(A \cap B^c) + P(B \cap A^c) \\ &= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] \\ &= P(A) + P(B) - 2P(A \cap B) \end{aligned}$$

(3) " At least one of A or B" is $A \cup B$ (4)" At most one of A or B" is $(A \cap B)^c$

$$P((A \cap B)^c) = 1 - P(A \cap B)$$

2. Let S be a sample space.
(1) Show that the collection $\mathbb{B} = \{\varnothing, S\}$ is a sigma algebra;
(2) Let $\mathbb{B}$ ={all subsets of S, including S itself}. Show that $\mathbb{B}$ is a sigma algebra;
(3) Show that the intersection of two sigma algebra is a sigma algebra.

Solution:
(1) The empty set $\emptyset \in \{\emptyset, S\} = B$ $\emptyset^c = S \in B$; $S^c = \emptyset \in B$ $\emptyset \in B$ and $S \in B$, $\emptyset \cup S = S \in B$
(2) $\emptyset \in B$ $\forall$ $A \in B$, $A \in S$, $A^c \in S$, thus, $A \in B$ $\forall A_i$, $i = 1, 2, 3, \dots$, where $A_i \in B$, then by definition, $A_i \in S$, $\cup A_i \in S$, thus $\cup A_i \in B$
(3)Let B_1 and B_2 are two algebras. then

- $\because$ $\emptyset \in B_1$ and $\emptyset \in B_2$, so $\emptyset \in B_1 \cap B_2$.
- $\forall A \in B_1 \cap B2$, $A \in B_1$, $A^c \in B_1$; $A \in B_2$, $A^c \in B_2$, thus $A^c \in B_1 \cap B_2$.
- $\forall A_{i=1,2,\dots} \in B_1 \cap B_2$, $A_{i=1,2,\dots} \in B_1$ and $\cup A_i \in B_1$; $\cup A_i \in B_2$; so $\cup A_i \in B_1 \cap B_2$.

3. Consider two events A and B such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{2}$. Determine the value of $P(B \cap A^c)$ for each of the following conditions: (1) A and B are disjoint; (2) $A \subset B$; (c) $P(A \cap B) = \frac{1}{8}$.

Solution:

- A and B are disjoint

$$P(B \cap A^c) = P(B) = \frac{1}{2}$$

- $A \subset B$

$$P(B \cap A^c) = P(B) - P(A \cap B)$$

Since $A \subset B$, $A \subset B$, then $P(A \subset B) = P(A)$. Therefore, $P(B \cap A^c) = P(B) - P(A) = \frac{1}{6}$.

- $P(A \cap B) = 1/8$

$$P(B \cap A^c) = P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

4. Let A and B be two events. Check if the following relations hold:
(1) $A \cup B = A \cup (A^c \cap B)$;
(2) $B = (A \cap B) \cup (A^c \cap B)$.
Please give your reasoning clearly.

Solution:

(1)By distributivity law,

$$\begin{aligned} A \cup (A^c \cap B) &= (A \cup A^c) \cap (A \cup B) \\ &= S \cap (A \cup B) \\ &= (A \cup B) \end{aligned}$$

(2) By distributivity law,

$$\begin{aligned} (A \cap B) \cup (A^c \cap B) &= (A \cup B) \cap B \\ &= S \cap B \\ &= B \end{aligned}$$

5. Suppose events A and B are mutually exclusive.
(1) Can we say that A^c and B^c mutually exclusive? Give your reasoning;
(2) Please find some example such that A^c and B^c are mutually exclusive as well.

Solution:

- If and only if $A \cup B = S$. Under this condition, we have

$$A^c \cap B^c = (A \cup B)^c = S^c = \emptyset$$

- Example:
Suppose $S = \{1, 2, 3\}$, $A = \{1\}$, $B = \{2, 3\}$.
A and B are mutually exclusive, so as A^c and B^c.

6. Suppose $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$. Is it possible that A and B are mutually exclusive?

Solution:

No. To prove this, we need to show that $P(A \cap B) > 0$

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= \frac{1}{3} + \frac{3}{4} - P(A \cup B) \\ &= \frac{13}{12} - P(A \cup B) \end{aligned}$$

Since $P(A \cup B) \leq 1$, we must have $P(A \cap B) > 0$

7. If 50% of the families in a city subscribe to the morning newspaper, 65% of the families subscribe to the afternoon newspaper, and 85% of the families subscribe to at least one of the two newspapers, what is the proportion of the families subscribe to both newspapers?

Solution:

Let $A = \{$families in a city subscribe to the morning newspaper$\}$, $B = \{$famililies subscribe to the afternoon n
The proportion of the families subscribe to both newspaper is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 50\% + 65\% - 85\% \\ &= 30\% \end{aligned}$$

8. Let A, B, C be any three events defined on sample space S. Express $P(A \cup B \cup C)$ in terms of $P(A), P(B), P(C), P(A \cap B), P(B \cap C), P(C \cap A)$ and $P(A \cap B \cap C)$. Give your reasoning.

Solution:

Let $D = B \cup C$

$$\begin{aligned} P(A \cup B) &= P(A) + P(D) - P(A \cap D) \\ &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) \\ &- (P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

9. A secretary types four letters to four people and addresses the four envelopes. If he inserts the letters at random, one into each envelope, what is the probability that exactly two letters will go into the correct envelopes? Exactly three?

Solution:

- Total number of possible outcomes is $4! = 24$. Number of selections of 2 correctly placed envelops is $C_4^2 = 6$. In the 2 correctly placed envelops, there is only one possible way to assign letters, i.e. the correct way. Thus, the number of 2 letters assigned is 6. $P(\text{exact 2 correctly assigned}) = \frac{6}{24} = \frac{1}{4}$
- $P(\text{ Exact three}) = 0$

10. An elevator in a building starts with five passengers and stops at seven floors. If each passenger is equally likely to get off at any floor and all passengers leave independently of each other, what is the probability that no two passengers will get off at the same floor?

Solution:

Assuming that the first passenger gets off at one of the seven floors, the second passenger must get off at one of the remaining six floors in which the passenger 1 does not get off and the probability of this case is 6/7. The third passenger must get off at one of the remaining five floors and the probability is 5/7. Repeating the process up to the fifth passenger, we get the probability:

$$\frac{7}{7} \times \frac{6}{7} \times \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = 0.1499$$

11. Suppose that a box contains r red balls and w white balls. Suppose also that balls are drawn from the box one at a time at random, without replacement.

(1) What is the probability that all r red balls will be obtained before any white balls are obtained?

(2) What is the probability that all r red balls will be obtained before two white balls are obtained?
Solution:

- It is equivalent to ask the probability of the first r balls obtained are red. So the number of ways to do that is P_r^r. The number of total ways to obtain r balls is P_{r+w}^r. So the probability is $\frac{P_r^r}{P_{r+w}^r} = \frac{r!w!}{(r+w)!} = \frac{1}{C_{r+w}^r}$.
- It is equivalent to ask the probability of obtaining r red balls and 1 white balls in the first (1+r) draws. So the number of ways to do it is $P_r^r C_w^1 C_{r+1}^1$ and the number of total ways is P_{r+w}^{r+1}. So the probability is $\frac{(r+1)!w!}{(r+w)!}$

12. Prove each of the following statements. (Assume that any conditioning event has positive probability.)
(1) If $P(B) = 1$, then $P(A \cap B) = P(A)$ for any A;
(2) If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$;
(3) If A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

Solution:
(1) $P(A) = P(A \cap B) + P(A \cap B^c). A \cap B^c \subset B^c$ and $P(B^c) = 0$. Therefore, $P(A \cap B^c) = 0$. Thus, $P(A) = P(A \cap B)$.
(2) $A \subset B$ implies that $A \cap B = A$. Thus,,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1.$$

and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

(3) A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$ and $A \cap (A \cup B) = A$. Thus,

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B)}$$

13. Suppose that 5% of men and 0.25% of women are color-blind. A person is chosen at random and that person is color-blind. What is the probability that the person is a male? (Assume males females to be in equal numbers.)
Solution:
Let M denotes Male, F denotes Female, CB denotes Color-Blinded. Using Bayes rule,

$$P(M|CB) = \frac{P(CB|M)P(M)}{P(CB|M)P(M) + P(CB|F)P(F)} = \frac{0.05 * 0.5}{0.05 * 0.5 + 0.0025 * 0.5} = 0.952$$

14. In a large city, it is established that 0.5% of the population has contracted AIDS. The available tests give the correct diagnosis for 80% of healthy persons and for 98% of sick persons. Suppose a

person is tested and found sick. Find the probability that the diagnosis is wrong, that is, that the person is actually healthy.

Solution:

Define $A = \{\text{Affected}\}$, $B = \{Healthy\}$

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A|B^c)P(B^c) + P(A|B)P(B)} \\ &= \frac{0.2 * 0.995}{0.2 * 0.995 + 0.98 * 0.005} \\ &= 0.976 \end{aligned} \tag{1}$$

15. If the probability of hinting a target is $\frac{1}{5}$, and ten shots are fired independently, what is the probability of the target being hit at least twice? What is the conditional probability that the target is hit at least twice, given that it is hit at least once?

Solution:

(1) $P(\#\text{ of hitting target} \geq 2) = \sum_{k=2}^{10} \binom{10}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{10-k} = 1 - \left(\frac{4}{5}\right)^{10} - 10\left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^9 = 0.624$

(2)

$$\begin{aligned} P(HT \quad &\geq \quad 2|HT \geq 1) = \frac{P((HT \geq 2) \cap (HT \geq 1))}{P(HT \geq 1)} = \frac{P(HT \geq 2)}{P(HT \geq 1)} \\ &= \quad \frac{1 - \left(\frac{4}{5}\right)^{10} - 10\left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^9}{1 - \left(\frac{4}{5}\right)^{10}} = 0.699 \end{aligned}$$

16. Standardized tests provide an interesting application of probability theory. Suppose first that a test consists of 20 multiple-choice questions, each with 4 possible answers. If the student guess on each question, then the taking of the exam can be modeled as a sequence of 20 independent events. Find the probability that the student gets at least 10 questions correct, given that he is guessing.

Solution:

$$\text{Pr} = \sum_{k=10}^{20} \binom{20}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{20-k} = 0.01386$$

17. Police plan to enforce speed limits by using radus traps at 4 different locations within the city limits. The radar traps at each of the locations L_1, L_2, L_3, and L_4 are operated 40%, 30%, 20%, and 30% of the time. and if a person who is speeding on his way to work has probabilities of passing through these locations, what is the probability that he will receive a speeding ticket? Give your reasoning.

Solution:

$P(A) = 0.8$. $P(B) = 0.9$. A and B are independent, thus $P(A \cap B) = P(A)P(B)$. The probability that the target can be hit is:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) \\ &= 0.8 + 0.9 - 0.8 * 0.9 = 0.98 \end{aligned}$$

18. Two events A and B are independent, and $B \subset A$. Find $P(A)$.

Solution:

$B \subset A \rightarrow P(A \cap B) = P(B)$. $\because$ A is independent of B, then we have $P(B) = P(AB) =$

$P(A)P(B)$.

$$P(B)(1 - P(A)) = 0 \tag{2}$$

So either P(B)=0, $0 \leq P(A) \leq 1$, or P(A)=1.

19. Suppose $0 < P(B) < 1$. Show that event A and B are independent if and only if $P(A|B) = P(A|B^c)$.

Solution:

(1) "If": If $P(A|B) = P(A|B^c)$, $P(A) = P(A \cap B) + P(A \cap B^c) = P(B)P(A|B) + P(B^c)P(A|B^c) = P(B)P(A|B) + P(B^c)P(A|B) = (P(B) + P(B^c))P(A|B) = P(A|B)$. Thus A and B are independent.

(2) "Only if": If A and B are independent, A and B^c are also independent. $P(A|B) = P(A) = P(A|B^c)$

20. Which of the following statements is true? If it is true, prove it; otherwise, give a counter example.

(1) If $P(A) + P(B) + P(C) = 1$, then the events A, B, C are mutually exclusive;

(2) If $P(A \cup B \cup C) = 1$, then A, B, C are mutually exclusive.

Solution:

(1)False. Let $S = \{1, 2, 3, 4\}$ and $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \emptyset$.

(2) False. Example: $A = B, C = A^c$.

21. Let $A_i, i = 1, 2, ..., n$, be a sequence of events of an experiment, where n is an integer. Show that $P\left(\cup_{i=1}^n A_i\right) \leq \Sigma_{i=1}^n P(A_i)$.

Solution:

$$\begin{aligned} P(\cup_{i=1}^n A_i) &= P(A_1 \cup (\cup_{i=2}^n A_i)) \\ &= P(A_1) + P(\cup_{i=2}^n A_i) - P(A_1 \cap (\cup_{i=2}^n A_i)) \\ &\leq P(A_1) + P(\cup_{i=2}^n A_i) \end{aligned}$$

Repeart this inequality for n times, we get the desired results.