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Probability and Statistics for Economists
Chapter 7 Convergences and Limit Theorems

1. Suppose $X_1, X_2, ...$ is an uncorrelated sequence with $E(X_i) = \mu$, $\text{var}(X_i) = \sigma_i^2$, and $\lim_{n\to\infty} \Sigma_{i=1}^n \sigma_i^2/i^2 = 0$. Show $\bar{X}_n$ converges to μ in quadratic mean.

Solution:

By definition we need to calculate $E(\bar{X}_n - \mu)^2$ and check if it converges to 0 as $n \to \infty$.

$$\begin{aligned} E(\bar{X}_n - \mu)^2 &= E\left[\frac{1}{n}\sum_{i=1}^n (X_i - \mu)\right]^2 \\ &= \frac{1}{n^2}\sum_{i=1}^n Var(X_i) \\ &= \sum_{i=1}^n \frac{\sigma_i^2}{n^2} \\ &\le \sum_{i=1}^n \frac{\sigma_i^2}{i^2} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus, we have showed $\bar{X}_n$ converges to μ in quadratic mean.

2. Suppose $(X_1, ..., X_n)$ is an IID random sample from the $N(0, \sigma^2)$ population, where $0 < \sigma^2 < \infty$. Define the sample mean $Z_n = n^{-1}\sum_{i=1}^n X_i$.

(1) Find the sampling distribution distribution $F_n(z)$ of Z_n for each $n \ge 1$;

(2) Find the limiting distribution of Z_n as $n \to \infty$;

(3) Is the limiting distribution of Z_n the same as $\lim_{n\to\infty} F_n(z)$? Explain.

(4) Find the limiting distribution of $\sqrt{n}Z_n$ as $n \to \infty$.

Solution:

Given Z_n is the sample mean, we have

(1) for each finite $n \ge 1$, Z_n is the mean of the sum of n independent normal random variables, then by reproductive property of normal distribution, we have

$$n * Z_n \sim N(0, n\sigma^2).$$

Thus, we have $Z_n \sim N(0, \frac{\sigma^2}{n})$ for any $n \ge 1$.

(2) As $n \to \infty$,we have the variance of Z_n converges to 0. Then we have the limiting distribution of Z_n become degenerate at point 0, which is the population mean. Then the limiting distribution of Z_n is given by $F_Z(z)$:

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0 \\ 1, & \text{if } z \ge 0 \end{cases}$$

(3) First, write down the CDF of Z_n:

$$\begin{aligned} F_{Z_n}(z) &= \int_{-\infty}^{z} \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2 \times n}{2\sigma^2}} du \\ &= \int_{-\infty}^{z\sqrt{n}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{v^2}{2\sigma^2}} dv, \end{aligned}$$

where the last equality comes from change of variable. Then we know as $n \to \infty$,

$$F_{Z_n}(z) = \begin{cases} 0, \text{ if } z < 0 \\ 1/2, \text{ if } z = 0 \\ 1, \text{ if } z > 0 \end{cases}$$

The limiting distribution of Z_n is not the same as $lim_{n\to\infty} F_{Z_n}(z)$. See page 21 in lecture note 7 for explanation.

(4) Since $Z_n \sim N(0, \frac{\sigma^2}{n})$, we know $\sqrt{n}Z_n \sim N(0, \sigma^2)$. There is no sample parameter in the CDF function of $\sqrt{n}Z_n$, we expect the limiting distribution doesn't change with the sample size, and it still follows a normal distribution with mean 0 and variance σ^2.

3. Suppose $(X_1, ..., X_n)$ is an IID random sample from the uniform distribution $U[\theta, 1]$, where $\theta < 1$. Define an estimator for θ as $Z_n = \min_{1\le i\le n} X_i$.

(1) Show that Z_n is consistent for θ as $n \to \infty$;

(2) Find the limiting distribution of $n(Z_n - \theta)$ as $n \to \infty$.

Solution:

(1) To show Z_n is consistent for θ, we just need to show that Z_n converges to θ in probability. For all $\varepsilon > 0$, we need

$$\lim_{n\to\infty} P(|Z_n - \theta| > \varepsilon) = 0$$

By definition of Z_n, the above probability can be written as

$$\begin{aligned} & P(Z_n > \theta + \varepsilon) \\ = \; & P(Z_i > \theta + \varepsilon) \quad \text{ for all } i \\ = \; & \prod_{i=1}^{n} P(Z_i > \theta + \varepsilon) \\ = \; & \prod_{i=1}^{n} (\frac{1-\theta-\varepsilon}{1-\theta}) \\ = \; & \left(\frac{1-\theta-\varepsilon}{1-\theta}\right)^n \\ \to \; & 0 \;\text{ as } n \to \infty \end{aligned}$$

(2) To find the limiting distribution of $n(Z_n - \theta)$, let's work on its CDF:

$$\begin{aligned}
& P(n(Z_n-\theta) \le z) \\
= \;& P(Z_n - \theta \le \frac{z}{n}) \\
= \;& P(Z_n \le \frac{z}{n} + \theta) \\
= \;& 1 - P(Z_n > \frac{z}{n} + \theta) \\
= \;& 1 - \left(\frac{1-\theta-\frac{z}{n}}{1-\theta}\right)^n \\
= \;& 1 - \left(1 - \frac{z}{n(1-\theta)}\right)^n \\
\to \;& 1 - e^{-z/(1-\theta)}
\end{aligned}$$

Thus we know $n(Z_n - \theta)$ converges in distribution to an exponential distribution with rate parameter to be $1-\theta$.

4. Suppose a sequence of random variables $\{Z_n\}$ is defined as

Z_n	$\frac{1}{n}$	n
P_{Z_n}	$1-\frac{1}{n}$	$\frac{1}{n}$

(1) Does Z_n converge in mean square to 0? Give your reasoning clearly;

(2) Does Z_n converge in probability to 0? Give your reasoning clearly;

(3) Does Z_n converge almost surely to 0? Give your reasoning clearly.

Solution:

(1) No. See Example 7.17 on page 376 of textbook .

(2) Yes. See Example 7.17 on page 376 of textbook .

(3) Not necessarily. Since we don't know the probability measure of the basic outcomes that maps Z_n to n. Compare this to the two examples discussed in the section.

5. Define $X_n = Y_n + Z_n$, where $\{Y_n\}$ is an IID sequence from a $N(0,1)$ population, $\{Z_n\}$ follows the sequence of distributions stated Exercise #4, and X_n and Y_n are mutually independent.

(1) Find the limiting distribution of X_n. Show your reasoning;

(2) The limiting distribution is also called the asymptotic distribution, and the mean and variance of the asymptotic distribution are called the asymptotic mean and asymptotic variance respectively. Find $\lim_{n\to\infty} E(X_n)$ and $\lim_{n\to\infty}\text{var}(X_n)$. Are they the same as the asymptotic mean and asymptotic variance respectively? Show your reasoning.

Solution:

(1) Given Y_n is IID normal random variable and the index n has no impact on the limiting distribution of Y_n. We know as $n \to \infty$, $Y_n \to N(0,1)$. While for Z_n, we have shown that $Z_n \xrightarrow{p} 0$. Then by Slutsky's Theorem, we have $X_n = Y_n + Z_n \xrightarrow{d} N(0,1)$.

(2) From part (1), we have shown that the limiting distribution of X_n is $N(0,1)$, then its asymptotic mean is 0 and asymptotic variance is 1. Let's compare them to $\lim_{n\to\infty}E(X_n)$ and $\lim_{n\to\infty}var(X_n)$.

$$\begin{aligned} E(X_n) &= E(Y_n) + E(Z_n) \\ &= 0 + \frac{1}{n} \times (1 - \frac{1}{n}) + n \times \frac{1}{n} \\ &= 1 + \frac{1}{n} - \frac{1}{n^2} \end{aligned}$$

Then we know $\lim_{n\to\infty}E(X_n) = 1$.

$$\begin{aligned} var(X_n) &= var(Y_n) + E(Z_n) \\ &= 1 + \frac{1}{n^2} \times (1 - \frac{1}{n}) + n^2 \times \frac{1}{n} \\ &= 1 + \frac{1}{n^2} - \frac{1}{n^3} + n \end{aligned}$$

Then we know $\lim_{n\to\infty}var(X_n) = \infty$.
Apparently, the asymptotic mean and asymptotic variance are different from $\lim_{n\to\infty}E(X_n)$ and $\lim_{n\to\infty}var(X_n)$.

6. Let the sample space S be the closed interval [0,1] with the uniform probability distribution. Define $Z(s) = s$ for all $s \in [0,1]$. Also, for $n = 1, 2, ...$, define a sequence of random variables

$$Z_n(s) = \begin{cases} s + s^n & \text{if } s \in [0, 1 - n^{-1}] \\ s + 1 & \text{if } s \in (1 - n^{-1}, 1]. \end{cases}$$

(1) Does Z_n converge in quadratic mean to Z?
(2) Does Z_n converge in probability to Z?
(3) Does Z_n converge almost surely to Z?

Solution:

See Example 7.21 on page 381 of textbook.

7. Suppose $g(\cdot)$ is a real-valued continuous function, and $\{Z_n, n = 1, 2, ...\}$ is a sequence of real-valued random variables which converges in probability to random variable Z. Show $g(Z_n) \to^p g(Z)$.

Solution:

See Lemma 7.5 on page 376 of textbook.

8. Suppose $g(\cdot)$ is a real-valued continuous function, and $\{Z_n, n = 1, 2, ...\}$ is a sequence of real-valued random variables which converges almost surely to random variable Z as $n \to \infty$. Show $g(Z_n) \to^{a.s.} g(Z)$.

Solution:

See Lemma 7.7 on page 384 of textbook.

9. Let $Z_1, Z_2, ...$ be a sequence of random variables that converges in probability to a constant a. Assume that $P(Z_n > 0) = 1$ for all n.

(1) Verify that the sequences defined by $Y_n = \sqrt{Z_n}$ and $X_n = a/Z_n$ converge in probability respectively.

(2) Use the results in part (a) to prove that σ/S_n convergences in probability to 1, assuming that S_n^2 converges to σ^2 in probability.

Solution:

(1) First we show that $Y_n = \sqrt{Z_n}$ converges to $\sqrt{a}$ in probability. Let $g(\cdot) = \sqrt{\cdot}$. Then we can see that $g(\cdot)$ is continuous. Then following the result in question 7, we immediately have the result that $Y_n \xrightarrow{p} \sqrt{a}$.

Next, we show that $X_n = a/Z_n$ converges to 1 in probability. For any $\varepsilon > 0$,

$$\begin{aligned}
& P(|a/Z_n - 1| < \varepsilon) \\
= \; & P(1 - \varepsilon < a/Z_n < 1 + \varepsilon) \\
= \; & P(\frac{a}{1+\varepsilon} < Z_n < \frac{a}{1-\varepsilon}) \\
= \; & P(\frac{a}{1+\varepsilon} - a < Z_n - a < \frac{a}{1-\varepsilon} - a) \\
= \; & P(\frac{-a\varepsilon}{1+\varepsilon} < Z_n - a < \frac{a\varepsilon}{1-\varepsilon}) \\
\geq \; & P(\frac{-a\varepsilon}{1+\varepsilon} < Z_n - a < \frac{a\varepsilon}{1+\varepsilon}) \\
= \; & P(|Z_n - a| < \frac{a\varepsilon}{1+\varepsilon}) \\
\to \; & 1 \quad \text{as } n \to \infty
\end{aligned}$$

Given $Z_n \xrightarrow{p} a$, then we know for any $\eta > 0$, we can find a $\delta > 0$ an $N > 0$ such that for all $n > N$, $P(|Z_n - a| > \eta) < \delta$. The last equality is achieves by letting $\eta = \frac{a\varepsilon}{1+\varepsilon}$. Thus we have $a/Z_n \xrightarrow{p} 1$.

(2) Let $a = \sigma$, $Z_n = S_n^2$, by $S_n^2 \xrightarrow{p} \sigma^2$, we have $S_n = \sqrt{S_n^2} \xrightarrow{p} \sigma$. And $\sigma/S_n \xrightarrow{p} 1$.

10. [*Delta Method*] Suppose $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} N(0, 1)$ as $n \to \infty$, and function $g(\cdot)$ is continuously differentiable with $g'(\mu) \neq 0$. Then show that as $n \to \infty$,

$$\sqrt{n}\left[g(\bar{X}_n) - g(\mu)\right] \xrightarrow{d} N(0, \sigma^2[g'(\mu)]^2)$$

and equivalently,

$$\frac{\sqrt{n}\left[g(\bar{X}_n) - g(\mu)\right]}{\sigma g'(\mu)} \xrightarrow{d} N(0, 1).$$

Solution:

See Lemma 7.11 on page 404 of textbook.

11. Suppose that $\{Z_1, \cdots, Z_n\}$ is an IID $N(0,1)$ random sample. What is the limiting distribution of $(\sum_{i=1}^n Z_i^2 - n)/\sqrt{n}$? Give your reasoning clearly.

Solution:

Easy to see that $\{Z_i^2\}$ follow IID χ_1^2. Then by CLT, $\frac{\sum_{i=1}^n Z_i^2 - E(\sum_{i=1}^n Z_i^2)}{\sqrt{var(\sum_{i=1}^n Z_i^2))}} \xrightarrow{d} N(0,1)$. Therefore, $\frac{\sum_{i=1}^n Z_i^2 - n}{\sqrt{2n}} \xrightarrow{d} N(0,1)$. Then $\frac{\sum_{i=1}^n Z_i^2 - n}{\sqrt{2n}} \xrightarrow{d} N(0,2)$.

12. Suppose $\mathbf{X}^n$ is an IID random sample from a population with $E(X_i) = \mu$, $\text{var}(X_i) = \sigma^2, E[(X_i - \mu)^4] = \mu_4$. Define $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$.

(1) Show that $S_n^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$;

(2) Derive the limiting distribution of $\sqrt{n}(S_n^2 - \sigma^2)$ as $n \to \infty$.

Give your reasoning.

Solution:

(1)

$$\begin{aligned} E(S_n^2 - \sigma^2)^2 &= var(S_n^2) \\ &= \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}(\sigma^2)^2\right] \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for the last equality above, see Section Notes 8 for details. Since $\xrightarrow{q.m.}$ implies $\xrightarrow{p}$, the result follows.

(2) CLT shows that $\frac{S_n^2 - E(S_n^2)}{\sqrt{var(S_n^2)}} = \frac{\sqrt{n}(S_n^2 - \sigma^2)}{\sqrt{\mu_4 - \frac{n-3}{n-1}\sigma^4}} \xrightarrow{d} N(0,1)$. Therefore, $\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{d} N(0, \mu_4 - \sigma^4)$.

Note: part (2) is questionable since the CLT here is note Lindeberg-Levy's since the sequence is not IID in S_n^2.

13. Suppose $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} N(0,1)$, where $-\infty < \mu < \infty$ and $0 < \sigma < \infty$. Find a nondegenerate limiting distribution of a suitably normalized version of the following statistics:

(1) $Y_n = e^{-\bar{X}_n}$;

(2) $Y_n = \bar{X}_n^2$, where $\mu = 0$ in this case.

Give your reasoning.

Solution:

(1) Let $g(x) = e^{-x}$. Clearly $g'(\mu) = -e^{-\mu} \neq 0$. Then by (1st order) Delta Method, $\frac{\sqrt{n}[Y_n - e^{-\mu}]}{-\sigma e^{-\mu}} \xrightarrow{d} N(0,1)$.

(2) Let $g(x) = x^2$. Clearly $g'(\mu) = 2\mu = 0$, but $g''(\mu) = 2 \neq 0$. Therefore, we can use Second Order Delta Method to get $\frac{n[Y_n - \mu^2]}{\sigma^2} \xrightarrow{d} \chi_1^2$.